Yablo's paradox and ω-inconsistency

Shunsuke Yatabe

Graduate School of Letters, Kyoto University, Japan
shunsuke.yatabe@gmail.com

Numbers and Truth
20 October 2012, Gothenburg

Outline

Theme: a syntactical nature of Yablo's paradox

- how Yablo propositions, paradoxical sentences in Yablo's paradox, are constructed: **coinductive construction**

Testbed: **ω-inconsistency**

- Yablo propositions have been constructed in ω-inconsistent truth theories,
- is ω-inconsistency necessary to construct Yablo propositions?

Results: working in **ZFA** and comparing that to the results in truth theories

- we can code Yablo propositions by using hypersets though **ZFA is ω-consistent**,
- **ω-consistency:**
  - **ZFA** allows **coinduction**,
  - truth theories only allows the **mixture of induction and coninduction**.

Yablo's Paradox

Let us assume there exist infinitely many propositions $\langle S_0, S_1, S_2, \cdots \rangle$ such that

S_n insists that **S_i is false for any $i > n$**

they imply a contradiction in classical logic.

- If S_0 is false,
  - there must be $j > 0$ such that S_j is true, so all $S_{j+1}, S_{j+2}, S_{j+3}, \cdots, S_k, \cdots$ must be false,
  - however, if S_{j+1} is false, then there exists $k > j + 1$ such that S_k is true, a contradiction.
- If S_0 is true, then $S_1, S_2, \cdots$ are false, identical to the previous case.

Motivation (1): providing a comparison example

- **A source of trouble:**
  only one method of constructing $\langle S_n : n \in \omega \rangle$ is known, using **diagonalization** in a truth theory [P97].
- **Problem:**
  The lack of comparison examples could lead to a **misunderstanding**: we might regard properties that **contingently** hold in the truth theory (and do not hold in other theories) as **essential** properties of Yablo's paradox.
- **Example:**
  - consistent **truth theories** with sufficient expressive power should be **ω-inconsistent** [L01],
  - but we do not know **whether ω-inconsistency is essential** in Yablo's paradox.

Motivation (2): Coinductive construction

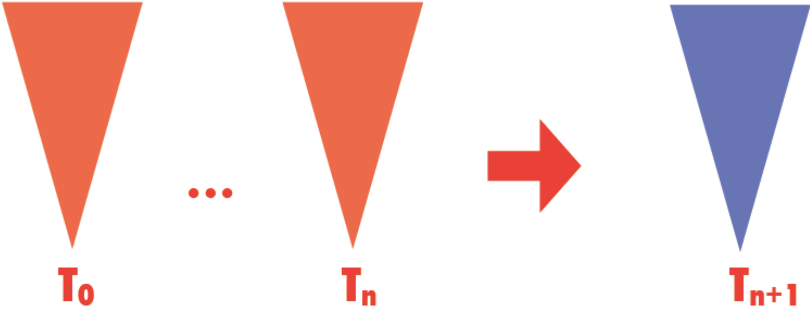
Yablo's paradox is important b/c it is an interesting example of coinductive construction:

- A characteristic property of Yablo propositions:
  - **each S_i is constructed by directly using S_{i+1} and S_{i+2}**: S_i is intuitively $\bigwedge_{j>i} \neg \mathbf{Tr}(\ulcorner S_j \urcorner)$: therefore,

$$\begin{aligned} S_i &\equiv \neg \mathbf{Tr}(\ulcorner S_{i+1} \urcorner) \wedge S_{i+2} \\ \neg S_i &\equiv \mathbf{Tr}(\ulcorner S_{i+1} \urcorner) \vee \neg S_{i+2}. \end{aligned}$$

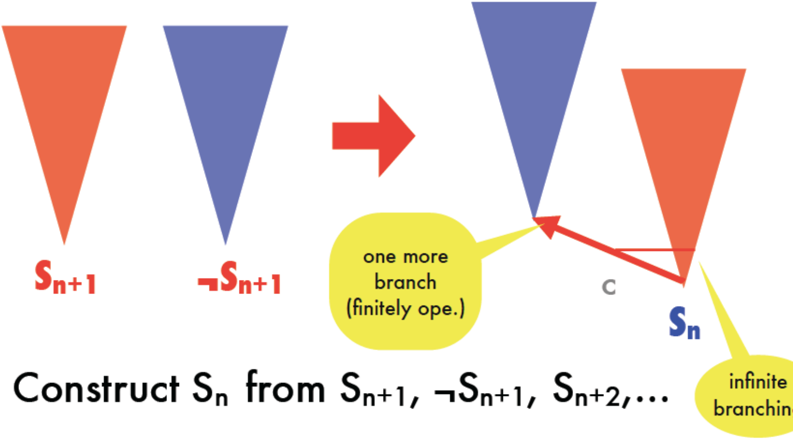
  - **infinite regress**; we need infinitely many $\langle S_{i+1}, S_{i+2}, S_{i+3}, \cdots \rangle$ to construct S_i in the end.
- Such constructions are called **coinductive**,
  - widely used in computer science to represent behaviors of non-terminate automatons [C93],
  - they are potentially infinite objects by finite constructions,

Induction



Constructing a tree T_{n+1} by using finitely many trees ($T_0, \cdots, T_n$) which already exist

Coinduction



Construct S_n from S_{n+1}, $\neg S_{n+1}$, $S_{n+2}, \ldots$

⇒Co-inductive construction

A coinductive language

We need **coinductive language** to write down Yablo propositions:

- One of the most famos coinductive language is **to use ZFA** [BE87] [BM96]
  - done by coding coinductively defined propositions by **hypersets**!
  - Yablo once suggested fixing **ZFA** as an analysis framework [Yab06], but abandoned this.
- The real significance of the framework of **the liar** [BE87] is
  - not to solve the liar paradox,
  - but **to provide a common framework of analyzing circular and co-inductive propositions!**

  We can code propositions, construct semantics, etc.

ZFA

ZFA is an axiomatic set theory, **ZF** minus the axiom of foundation plus the anti-foundation axiom (**AFA**).

- **flat system**:
  $\langle X, A, e\rangle$ is a flat system of equations if
  - $X \subseteq U$ (urelements, interpreted as variables),
  - A is an arbitrary set, and
  - $e : X \to \mathcal{P}(X \cup A)$.
- **Example:** $\langle\{a\}, \emptyset, \{\langle a, \{a\}\rangle\}\rangle$ represents the equation

  $$x = \{x\}$$

  where x is a free variable since $e(a) = \{a\}$.
- **Theorem: AFA** guarantees that **any flat system of equations defines hypersets uniquely**.
  This is a sort of coinductive definition: consider the equations $x_n = \{x_{n+1}, x_{n+2}\}$ for any $n \in \omega$!

Coding propositions in ZFA (original)

- Propositions are coinductively coded in **ZFA** as follows:
  - **infinitary conjunction**

$$\ulcorner \bigwedge_{i \in I} A_i \urcorner = \{\{\mathbf{c}, \ulcorner A_i \urcorner\} : i \in I\}$$

  - **infinitary disjunction**

$$\ulcorner \bigvee_{i \in I} A_i \urcorner = \{\{\mathbf{d}, \ulcorner A_i \urcorner\} : i \in I\}$$

  - $\ulcorner \neg A \urcorner = \{\mathbf{n}, \ulcorner A \urcorner\}$,
  - $\ulcorner \mathrm{Tr}(A) \urcorner = \{\mathbf{t}, \ulcorner A \urcorner\}$

  for some fixed set $\mathbf{c}, \mathbf{d}, \mathbf{n}, \mathbf{t}$.
- **Example:** the liar proposition is coded by λ satisfying

$$x = \{\mathbf{n}, \{\mathbf{t}, x\}\}$$

Coding Yablo propositions in ZFA (original)

Remember

$$S_i \quad \equiv \quad \wedge_{j \geq i+1} \neg \mathrm{Tr}(\ulcorner S_j \urcorner)$$

Therefore Yablo propositions $\{S_n : n \in \omega\}$ are coded by the following equation:

$$x_n \quad = \quad \{\{\mathrm{c}, \{\mathrm{n}, \{\mathrm{t}, x_k\}\}\} : k > n\}$$

Then $S_0, S_1, \cdots$ are solutions of $x_0, x_1, \cdots$.

Coding Yablo propositions in $\mathbf{ZFA}$ (More intuitive)

- **Truth predicate** is **not necessary** for simulating **the structure of derivations** in Yablo's paradox in $\mathbf{ZFA}$!
- Defining positive and negative propositions separately:
  - **positive propositions**: $Y_0, Y_1, \cdots$ are solutions of $x_0, x_1, \cdots$,
  - **negative propositions**: $\neg Y_0, \neg Y_1, \cdots$ are solutions of $y_0, y_1, \cdots$
- The equations are as follows:

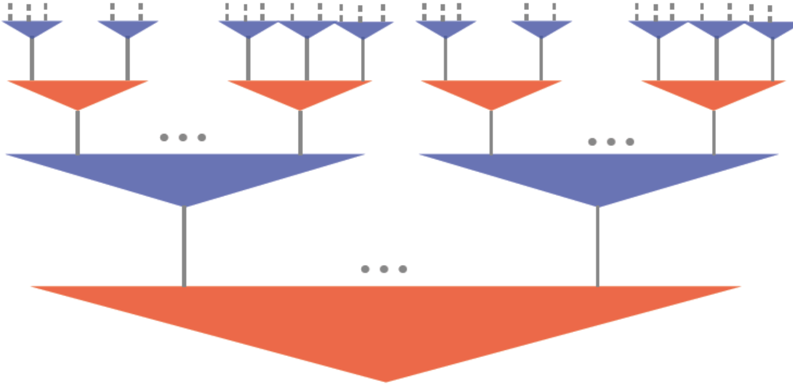
$$
\begin{aligned}
x_n &= \{\{\mathrm{c}, y_k\} : k > n\} \\
y_n &= \{\{\mathrm{d}, x_k\} : k > n\}
\end{aligned}
$$

- The intuitive meaning: $Y_n \equiv \wedge_{n<i} \neg Y_i$ and $\neg Y_n \equiv \vee_{n<i} Y_i$,

Remark: Yablo pointed out that they are identical in $\mathbf{ZFA}$, but adding indexes makes them pairwise different!

How Yablo trees look like?

Yablo propositions forms self-similar infinite branching tree of infinite height



Red: infinite conjunction,
Blue: infinite disjunction

What makes different on ω-consistency?

- **Known:**
  well-known consistent theories with sufficient expressive power are **ω-inconsistent**.
  - **Yablo paradox**,
  - **McGee's paradox** for $\mathbf{\Gamma}$ [Mc85] and $\mathbf{CT}_\omega$[HH05],
  - **Modest liar paradox** for $\mathbf{PA\text{Ł}Tr_2}$ [HPS00], etc.

- **What we have shown:**
  Yablo propositions can be constructed in $\mathbf{ZFA}$ though it is
  - consistent,
  - **ω-consistent**,

- **Problem:** what makes the difference?

The main difference

- **ZFA**: **coinduction**
  - the construction does not have initial case,
  - truth predicate is not necessary (**ZFA** already has a machinery for coinductive construction),
- **truth theories**: **the mixture of induction and coinduction** (coinductive construction with the initial case.)
  - have **an initial case**
  - truth predicate is necessary: it enables to apply the fixed point lemma (which generate potentially infinite propositions).

Potentially infinite propositions in truth theories

- Yablo proposition $\bar{S}_0$ is constructed by the fixed point lemma:

$$\bar{S}_0 \equiv (\forall z)[z > 0 \to \neg \underbrace{\mathbf{Sat}(\ulcorner \bar{S}_0 \urcorner, z)}_{\mathbf{Tr}(\bar{S}_z)}],$$

where $\mathbf{Sat}(\ulcorner \varphi(x) \urcorner, z) \equiv \mathbf{Tr}(\ulcorner \varphi(z) \urcorner)$.

- The intuitive meaning of paradoxical formulae:

Yablo: infinite sentence

$$\bar{S}_0 \equiv \neg\mathbf{Tr}(\bar{S}_1) \wedge (\neg\mathbf{Tr}(\bar{S}_2) \wedge (\neg\mathbf{Tr}(\bar{S}_3) \wedge \cdots)))$$

McGee: infinite sentence (nested)

$$\gamma \equiv \neg\mathbf{Tr}(\ulcorner \mathbf{Tr}(\ulcorner \mathbf{Tr}(\ulcorner \mathbf{Tr}(\ulcorner \cdots \mathbf{Tr}(\ulcorner \gamma \urcorner) \cdots \urcorner)\urcorner)\urcorner)\urcorner)$$

Modest liar: infinite sentence (nested)

$$A \equiv \mathbf{Tr}(\neg A \to (\neg A \to \cdots \to (\neg A \to A)\cdots))$$

The closer look at McGee's Paradox

ω-inconsistency is proved by γ in $\mathbf{\Gamma}$ [Mc85]:

$$\gamma \equiv \neg\forall x \mathbf{Tr}(f(x, \ulcorner\gamma\urcorner))$$

$$f(n, \ulcorner\varphi\urcorner) = \ulcorner\underbrace{\mathbf{Tr}(\ulcorner\cdots\mathbf{Tr}}_{n \text{ times}}(\ulcorner\varphi\urcorner)\cdots)\urcorner$$

γ is a limit of the following (finite) operations:

$$\gamma_0 \equiv \neg\mathbf{Tr}(\ulcorner\gamma\urcorner) \quad \cdots \text{the initial case!}$$

$$\gamma_1 \equiv \neg\mathbf{Tr}\ulcorner\mathbf{Tr}(\ulcorner\gamma\urcorner)\urcorner)$$

$$\gamma_2 \equiv \neg\mathbf{Tr}\ulcorner\mathbf{Tr}\ulcorner\mathbf{Tr}(\ulcorner\gamma\urcorner)\urcorner)\urcorner)$$

$$\vdots$$

$$\gamma \equiv \neg\underbrace{\mathbf{Tr}(\ulcorner\cdots(\ulcorner\mathbf{Tr}}_{\infty \text{ many}}(\ulcorner\gamma\urcorner)\urcorner)\cdots)\urcorner) \quad \cdots \text{the limit}$$

In **ZFA**,

$$x = \{\mathbf{n}, y\}$$

$$y = \{\mathbf{t}, y\}$$

The comparison

- The meaning of infinitary propositions is different from finite propositions: inductive evaluation does not work,
- if the infinitary proposition is defined as the limit of finite propositions,
  - taking the limit sometimes violates a property which is own by any finite propositions in the limit sequence:
  - γ is a limit of the following (finite) operations:

$$\gamma_0 \equiv \neg\mathbf{Tr}(\ulcorner\gamma\urcorner)$$
$$\gamma_1 \equiv \neg\mathbf{Tr}\ulcorner\mathbf{Tr}(\ulcorner\gamma\urcorner)\urcorner)$$
$$\vdots$$
$$\gamma \equiv \neg\underbrace{\mathbf{Tr}(\ulcorner\cdots(\ulcorner\mathbf{Tr}(}_{\infty \text{ many}}\ulcorner\gamma\urcorner)\urcorner)\cdots)\urcorner)$$

- In **ZFA**, coinductive propositions are **isolated** from finite propositions.

$$g = \{\mathbf{n}, \{\mathbf{t}, \{\mathbf{t}, \{\mathbf{t}, \cdots\}\}\}\}$$

Conclusion

Theme: how Yablo propositions are constructed: **coinductive construction**

Our results: we coded Yablo propositions by using hypersets though $\mathbf{ZFA}$ is ω-consistent,

- ω-inconsistency is not necessary,
- it is caused by the difference of the form of construction,
  - $\mathbf{ZFA}$ allows pure coinduction,
  - truth theories only allows the mixture of induction and coinduction.

Future task: **semantics**

- Barwise-Etchemendy [BE87] style (done),
- Game theoretic semantics?

- Jon Barwise. John Etchemendy. The Liar: An Essay in Truth and Circularity. Oxford University Press (1987)
- Jon Barwise, Lawrence Moss. Vicious Circles. CSLI publications (1996)
- Thierry Coquand. Infinite Objects in Type Theory. TYPES (1993) 62-78.
- Petr Hájek, Jeff B. Paris, John C. Shepherdson. " The Liar Paradox and Fuzzy Logic" Journal of Symbolic Logic, 65(1) (2000) 339-346.
- Volker Halbach and Leon Horsten. The deflationist's axioms for truth. in J.C. Beal and B. Armour-Garb (eds.) Deflationism and Paradox. Oxford Universoty Press.
- Hannes Leitgeb. Theories of truth which have no standard models. Studia Logica, 68 (2001) 69-87.
- Vann McGee. How truthlike can a predicate be? A negative result. Journal of Philosophical Logic, 17 (1985): 399-410.
- Lawrence S. Moss. Coalgebra and Circularity. Preprint (2008)

- Graham Priest. Yablo's paradox. Analysis 57 (1997) pp.236-242.
- Matteo Viale. The cumulative hierarchy and the constructible universe of ZFA. Mathematical Logic Quarterly, 2004, volume 50, no. 1, pp.99-103.
- Stephen Yablo. Paradox Without Self-Reference. Analysis 53 (1993) 2251-52.
- Stephen Yablo. Circularity and Paradox. Self-reference. in Bolander, Hendricks, Pedersen (eds.), Self-reference, Stanford: CSLI Publications.
- Shunsuke Yatabe. A constructive naive set theory and infinity. Preprint.
- Shunsuke Yatabe. Yablo's paradox, a coinductive language and its semantics. Preprint.