Empiricism, Probability, and Knowledge of Arithmetic

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The thesis that I want to examine today is this:

> arithmetical knowledge may be legitimately extended by confirmation just as it may be by proof.

This thesis is one component of a *broader empirical account of arithmetical knowledge*, according to which:

> subsequent to some appropriately empirical acquisition of knowledge of quantifier-free truths like $7 + 5 = 12$,
>
> one may legitimately extend this knowledge to knowledge of more complex arithmetical truths by way of confirmation.

My plan is to present an admittedly partial and preliminary defense of certain precisification of the thesis against 2 objections of the form: this kind of probability is *too close* to arithmetical truth.

Among notions of confirmation, I focus on probabilistic variants:

> Hypothesis h is *confirmed* by evidence e relative to background knowledge K if $P(h|e \,\&\, K) > P(h|K)$

Here P is a probability assignment, a function $P : \mathrm{Sent}(L) \rightarrow \mathbb{R}$ which satisfies, for all φ, ψ in the signature L of arithmetic:

(P1) $P(\varphi) \geq 0$,
(P2) $P(\varphi) = 1$ if $\models \varphi$,
(P3) $P(\varphi \vee \psi) = P(\varphi) + P(\psi)$ if $\models \neg(\varphi \,\&\, \psi)$

where $\models$ is the usual consequence relation from first-order logic.

In cases where $P(e|K)$ and $P(h|K)$ aren't zero or one, then we have that confirmation occurs if $h \,\&\, K \models e$.

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Two special cases of the thesis are then:

I. Let Robinson's Q be written $h \equiv \forall\, \overline{x}\; F(\overline{x})$ and consider evidence of form $e \equiv \bigwedge_{i=1}^{N} F(\overline{a}_i)$. Then one may justifiably infer from e to h on the basis of justification in e and h's being confirmed by e.

II. Let $h \equiv \forall\, x\; G(x)$ and $e \equiv [G(0) \,\&\, \forall\, n\, (G(n) \rightarrow G(n+1))]$. Then one may justifiably infer from e to h against $K \equiv$ Robinson's Q on the basis of justification in e, K and h's being confirmed by e relative to the background of K.

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The significance of the thesis thus resides in the fact that:

> it accounts for our knowledge of arithmetical axioms, using a source of justification which is routinely and efficaciously employed in other parts of our ordinary reasoning, thus serving to partially dissipate a skeptical concern about the possibility of mathematical knowledge.

Outline

In this setting, the conjunctive evidence $e \,\&\, \bigwedge_{i=1}^{n} F(d_i)$ often constitutes better evidence for the universal hypothesis $h \equiv \forall\, x\; Fx$ than does the evidence e alone.

One might think that such a situation ought not arise in mathematics. One explanation for this might be that it is somehow inimical to the character of mathematical justification. However, there is an alternative explanation for why such a situation does not arise in several traditional mathematical domains.

Namely, known indiscernibility with respect to many predicates F is a feature of many traditional geometrical domains. But, in this case, adding the evidence $\bigwedge_{i=1}^{n} F(d_i)$ does not make for better evidence for $h \equiv \forall\, x\; Fx$ since this addition is obviously equivalent to the desired conclusion, and hence inferior qua evidence.

In arithmetical reasoning, Σ^0_1-completeness of Robinson's Q inhibits confirmation of Π^0_1-statements by instances. So many instances do not confirm Goldbach's conjecture or consistency statements.

For, consider Π^0_1-sentence $h \equiv \forall\, x\, F(x)$, and consider potential evidence of the form $e \equiv \bigwedge_{i=1}^{N} F(a_i)$. If this evidence is true then it is derivable from Robinson's Q. If Robinson's Q is in the background knowledge K, then $e \,\&\, K$ will be equivalent to K.

In such a situation, we will *never* have that $P(h|e \,\&\, K) > P(h|K)$, and so we will never have probabilistic confirmation.

The approach requires an independent source of justification for the evidence, and so will be inapplicable in the setting where we don't have a grip on such an independent source. For instance, if you assign probabilities to sentences in the signature of valued fields or categories then you won't get confirmation unless you specify what counts as evidence for some large subclass of such sentences.

In the arithmetical case, I'm envisioning the independent source as being something empirical like perception, which allows me to verify independently perceptual or logical correlates of quantifier-free arithmetical statements like $7 + 5 = 12$. The approach thus requires further articulation and defense of such source qua source of justification.

The Approach Requires Specification of Probability Assignments,

The approach obviously requires the specification of some class of probability assignments. Today, I want to focus on 2 objections, which have the form:

> this kind of probability assignment, and hence this kind of confirmation, is *too close* to arithmetical truth.

The thought is that due to this conceptual proximity:

> if one had skeptical doubts that we could come to be justified in believing arithmetical truths, then one should have equally strong doubts that we could come to be justified in believing that such-and-such sentences had such-and-such probability.

Outline

A counting measure is a probability assignment of the form:

$$P(\varphi) = a_1 \cdot T_1(\varphi) + \cdots + a_1 \cdot T_n(\varphi)$$

where $a_1 + \cdots + a_n = 1$ and $T_1, \ldots, T_n$ are in $[K]$, the space of complete consistent extension of our background knowledge K.

It seems that appeal to these kinds of probability assignments can't dissipate a skeptical concerns about mathematical knowledge.

For, if $T_1, \ldots, T_n$ included true arithmetic $Th(\mathbb{N})$, then one would want to know what it is about our relationship to true arithmetic which grants it this preferred status.

While if not, then one would want to know what about $T_1, \ldots, T_n$ make them reliable indicators of truth. Why these and not others?

A counting measure P induces a measure $\widehat{P} : \mathrm{Borel}([K]) \to [0,1]$ on the Borel subsets of the space $[K]$ by:

$$\widehat{P}([\varphi]) = a_1 \cdot T_1(\varphi) + \cdots + a_1 \cdot T_n(\varphi)$$

If P is a counting measure, then notes the elementary consequence:

$$\widehat{P}(\{T_i\}) = \lim_{\ell \to \infty} \widehat{P}([\chi_{T_i \upharpoonright \ell}]) = a_i > 0$$

where $\varphi_1, \ldots, \varphi_n, \ldots$ is fixed enumeration of $\mathrm{Sent}(L)$ and

$$\sigma \mapsto \chi_\sigma \equiv \bigwedge_{\sigma(i)=1} \varphi_i \wedge \bigwedge_{\sigma(i)=0} \neg\varphi_i$$

An *atom* of a probability measure $\widehat{P} : [K] \to [0,1]$ is a theory T in the space $[K]$ such that $\widehat{P}(\{T\}) > 0$.

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A *continuous* probability measure $\widehat{P} : [K] \to [0,1]$ is one which has no atoms.

Since $[K]$ uncountable when K includes Robinson's Q, the Lebesgue measure on $[0,1]$ transfers to $[K]$. In this way, one obtains a hyperarithmetically definable probability assignment $P : \mathrm{Sent}(L) \to [0,1]$ such that $\widehat{P} : [K] \to [0,1]$ is continuous.

However, it's not obvious whether one can effectivize this proof in such a way that one could ensure that P is arithmetically definable.

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Ostensibly, being continuous is very hard to test since its official definition requires checking something about every theory.

But by using König's Lemma, not too difficult to see that a probability measure $\widehat{P} : [K] \to [0,1]$ is *continuous* if and only if

$$\forall \, \epsilon > 0 \, \exists \, \ell \, \forall \, |\sigma| = \ell \, \widehat{P}([\chi_\sigma]) < \epsilon$$

Consider the partial order, where the order is by extension:

$$\mathbb{P} = \{P : S \to \mathbb{Q} : S \text{ finite algebra} \subseteq \mathrm{Sent}(L) \ \& \ P \models P1\text{-}P3\}$$

This Π^0_1-partial order has the following Δ^0_2-dense sets:

$$\begin{aligned} D_n &= \{P \text{ in } \mathbb{P} : \mathrm{dom}(P) \text{ contains } \varphi_n\} \\ D_\epsilon &= \{P \text{ in } \mathbb{P} : \exists\, \ell\ \mathrm{dom}(P) \supseteq \{\varphi_1, \ldots, \varphi_\ell\} \ \& \ \forall\ |\sigma| = \ell\ P(\chi_\sigma) < \epsilon\} \end{aligned}$$

So there's Δ^0_2-generic $G \subseteq \mathbb{P}$ and its union is a Δ^0_2-definable $P : \mathrm{Sent}(L) \to [0,1]$ such that $\widehat{P} : [K] \to [0,1]$ is continuous.

Moral: there are at least some probability assignments which aren't counting measures and which are still computationally tractable.

Outline

This problem arises with the simple observation that probability assignments only have rules for the propositional connectives, but are applied here to sentences with quantifiers.

Initial impetus: shouldn't there be rules for the quantifiers as well?

One natural rule is the following version of countable additivity:

$$\omega\text{-additivity: } (\mathrm{P}\omega)\ P(\forall\, x\, \varphi(x)) = \lim_N P(\textstyle\bigwedge_{n=1}^{N} \varphi(s^n(0)))$$

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But ω-additivity forces the alignment of the true and the probable:

> Suppose that $0 < \epsilon < \frac{1}{2}$ is a low error threshold, and suppose further that P is an ω-additive probability assignment such that $P(K) > 1 - \epsilon$, where $K \equiv$ Robinson's Q. Then for all φ
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> $$(\omega, 0, s, +, \times) \models \varphi \iff P(\varphi) > 1 - \epsilon \qquad (*)$$

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Why is this a problem? Difficult to know things about this assignment. High priors entails low degree of confirmation.

What would a *satisfactory* solution to this problem look like?

Need reasons for thinking that we're are not committed to ω-additivity in the same way that we're are committed to the other probabilistic rules.

My solution is that the only probabilistic rules which we're committed to are those which follow from invulnerability to a Dutch Book.

This solution is satisfactory because: it turns out that ω-additivity does not follow from invulnerability, even when one allows infinite sequences of bets and sentences.

My solution is that the only probabilistic rules which we're committed to are those which follow from invulnerability to a Dutch Book.

> One is *invulnerable to a Dutch Book* if for any finite sequences of bets on any finite set of sentences, there is a some possible situation in which one does not suffer a net loss.
>
> Assuming that possible situations are suitably identified with complete consistent theories, then one traditionally proves that P1-P3 follow from invulnerability to a Dutch Book.

This solution is satisfactory because: it turns out that ω-additivity does not follow from invulnerability, even when one allows infinite sequences of bets and sentences.

Outline

The thesis that I wanted to examine today is this:

> arithmetical knowledge may be legitimately extended by (probabilistic) confirmation just as it may be by proof.

We looked at 2 objections of the form:

> this kind of probability assignment, and hence this kind of confirmation, is *too close* to arithmetical truth.

The first objection was that counting measures seem inexplicably attracted towards truth or potentially unreliable. The response was to show that there are some probability assignments which aren't counting measures and which are computationally tractable.

The second objection was that the probabilistic ω-rule will force alignment of truth and high probability. The response was to suggest that we're committed to only those probabilistic rules which follow from invulnerability.

Suppose P is an arithmetically definable probability assignment, and that background knowledge K includes Robinson's Q.

By the diagonal lemma, there is L-sentence λ such that

$$Q \vdash (\lambda \leftrightarrow P(\lambda) = 0)$$

Since K includes Robinson's Q, we have the following:

$$\lambda \implies P(\lambda) = P(P(\lambda) = 0) = 0 \quad (1)$$

$$\neg\lambda \ \&\ P(\lambda) = 1 \implies P(\lambda \ \&\ P(\lambda) = 0) = 1 \quad (2)$$

$$\neg\lambda \ \&\ P(\lambda) < 1 \implies P(\lambda | P(\lambda) = 0) = 1 > P(\lambda) \quad (3)$$

Each of these seems problematic, but (3) more so, since it casts doubt on the idea that in general confirmation is a source of justification. I do not presently know which of (1)-(3) are possible.