Degrees of Interpretability of Finitely Axiomatized Sequential Theories

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Example

We have ZF $\rhd$ (ZF + CH) and ZF $\rhd$ (ZF + $\neg$ CH).

So CH is independent of ZF but not stronger than ZF.

On the other hand ZF + "there is an inacessible cardinal" is stronger than ZF.

Varieties of Use

- to explicate intuitions of sameness of theories.
  E.g., mutual interpretability, bi-interpretability.
- to transfer metamathematical information from one theory to another.
  E.g., consistency, essential undecidability.
- to compare theories w.r.t. strength.
  Reverse Mathematics.
- to import conceptual resources of one theory into another. In mathematician's terms: to increase the number of ways of 'seeing things'.
  E.g., in the proof of the Incompleteness Theorems.
- to formulate coordinate-free versions of theorems like G2.
- to provide a philosophical reduction of ontologies.
  As a sui-genericist, I am skeptical about this one.
  A thing is what it is and not another thing.
  (Bishop Joseph Butler).

Strength

In this lecture we are interested in interpretability as a means to measure the *strength* of theories.

For example GB $\underset{\neq}{\rhd}$ ZF, ZF $\underset{\neq}{\rhd}$ PA, EA $\underset{\neq}{\rhd}$ S^1_2.

Translations

We introduce one-dimensional, one-piece, relative, non-identity-preserving interpretations without parameters.
We refrain from defining the various richer notions. They only play a minor role in this lecture.

A *relative translation* $\tau : \Sigma \to \Theta$ is a pair $\langle \delta, F \rangle$.

- δ is Θ-formula with one free variable v_0.
- F associates to R of Σ of arity n a Θ-formula $F(R)$ with variables among $v_0, \ldots, v_{n-1}$.

Induced extension mapping:

- $(R(y_0, \cdots, y_{n-1}))^\tau := F(R)(y_0, \cdots, y_{n-1})$;
- $(\cdot)^\tau$ commutes with propositional connectives;
- $(\forall y\, A)^\tau := \forall y\, (\delta(y) \to A^\tau)$;
- $(\exists y\, A)^\tau := \exists y\, (\delta(y) \wedge A^\tau)$.

Interpretations

An interpretation K is of the form $\langle U, \tau, V \rangle$, where $\tau : \Sigma_U \to \Sigma_V$ and for all U-sentences A, we have: $U \vdash A \Rightarrow V \vdash A^\tau$.

Equivalently, we can demand that for all axioms A of U, including the ones for identity, we have: $U \vdash A \Rightarrow V \vdash A^\tau$.

We write $K : V \rhd U$ or $K : U \lhd V$ for K is an interpretation of the form $\langle U, \tau, V \rangle$.

An interpretation $K : V \rhd U$ is *faithful* iff, for all sentences A in the language of U, $U \vdash A$ iff $V \vdash A^\tau$.

- $U \lhd V$, or $V \rhd U$ iff $\exists K\ K : V \rhd U$.
- $U \equiv V$ iff $U \lhd V$ and $V \lhd U$.

Operations

We have an identity interpretation and interpretations can be composed. So:

These operations lift to operations on interpretations. Thus:

- $U \lhd U$.
- If $U \lhd V$ and $V \lhd W$, then $U \lhd W$.

Supremum and Infimum

We consider the structure of degrees of interpretability for finitely axiomatized theories.

We can define the infimum of two finitely axiomatized theories A and B by $A \otimes B := A \vee B$, where the signature is the union of the signatures of A and B.

We define the supremum $A \oplus B$ of two theories by making their signatures disjoint adding two domain predicates $\triangle_A$ and $\triangle_B$. We take the union of the relativized versions of the theories. We also replace identity for each theory by a new binary predicate.

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Definition

Sequentiality (Pudlák, 1983) is an explication of the idea of *theory with coding*. More precisely: it contains the coding adequate for building partial satisfaction predicates corresponding to a complexity measure that counts e.g. depth of alternating quantifiers.

It is essentially richer than e.g. *theory with pairing* which does not have sequences of variable length.

Sequential theories are also relevant for the study of *extending a theory with an external satisfaction predicate*.

Definition

Sequential theories have a very simple definition. We call an interpretation *direct* if it is identity preserving and unrelativised.

A theory is *sequential* iff it directly interprets *Adjunctive Set Theory*, AS.

The theory AS is a one-sorted theory with a binary relation $\in$.

AS1 $\vdash \exists x \forall y \; y \notin x$,

AS2 $\vdash \forall x, y \exists z \forall u \, (u \in z \leftrightarrow (u \in x \vee u = y))$.

We can build an interpretation of e.g. $\mathrm{I}\Delta_0 + \Omega_1$ in any sequential theory by an elaborate bootstrap. Similarly we can develop a theory of sequences for all objects.

Examples

Examples of sequential theories are:

- Adjunctive Set Theory AS.
- PA^-, the theory of discretely ordered commutative semirings with a least element. This was recently shown by Emil Jeřábek.
- Buss' theory S^1_2 and bi-interpretable variants of it like a theory of strings due to Ferreira, and a theory of sets and numbers due to Zambella.
- Wilkie and Paris' theory $\mathrm{I}\Delta_0 + \Omega_1$.
- Elementary Arithmetic EA (aka Elementary Function Arithmetic EFA, or $I\Delta_0 + \exp$).
- PRA.
- $\mathrm{I}\Sigma^0_1$.
- Peano Arithmetic PA.
- ACA_0.
- ZF.

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Degree Structures

We study:

- $\mathbb{D}_{\text{all}}$: the degree structure of all finitely axiomatized theories.
- $\mathbb{D}_{\text{seq}}$: the degree structure of all finitely axiomatized sequential theories.
- $\mathbb{V}_A$: the degree structure of all finite extensions of A.

Mycielski, Pudlák and Stern (1990) and Friedman (2007) show that $\mathbb{D}_{\text{all}}$ is a distributive lattice, that it is dense with an infinite antichain between any A, B with $A \mathrel{\ntrianglelefteq} B$, etc.

Vítěslav Švejdar asked in 1978: suppose $\mathsf{Q} \lhd A$. Do we have suprema in $\mathbb{V}_A$?

Adding Sequences to a Theory

In a trivial way we can extend any theory to a sequential theory (expanding the signature). This gives us a functor $\mathsf{SEQ} : \mathbb{D}_{\mathsf{seq}} \to \mathbb{D}_{\mathsf{all}}$.

Suppose emb is the identical embedding functor of $\mathbb{D}_{\mathsf{seq}}$ into $\mathbb{D}_{\mathsf{all}}$. Let B be sequential. We have:

$$\mathsf{SEQ}(A) \lhd_{\mathsf{seq}} B \text{ iff } A \lhd \mathsf{emb}(B).$$

Thus, SEQ is the left adjoint of emb.

We can take the supremum of A and B in $\mathbb{D}_{\mathsf{seq}}$ to be

$$A \sqcup B := \mathsf{SEQ}(A \oplus B),$$

where $\oplus$ is the supremum in $\mathbb{D}_{\mathsf{all}}$. The infimum remains the same in both degree structures.

A Normal Form Theorem

Let ρ be depth of quantifier alternations. We write $\Box_{A,n}$ for provability from A involving only formulas B with $\rho(B) \leq n$.

Pudlák 1985:
Suppose A is finitely axiomatized and sequential. We have:

$$A \equiv (\mathsf{S}^1_2 + \Diamond_{A,\rho(A)}\top).$$

We could have taken Q, PA^- or, if you wish $\mathrm{I}\Delta_0 + \Omega_{17}$ here. S^1_2 has the advantage that it is finitely axiomatizable and that arithmetization of syntax works very naturally.

Connection to EA

Wilkie & Paris 1987:
For any Π^0_1-sentences P, P', we have:

$$(\mathsf{S}^1_2 + P) \rhd (\mathsf{S}^1_2 + P') \;\Leftrightarrow\; \mathsf{EA} \vdash P \rightarrow P'.$$

Friedman $\leq$ 1985:
Suppose A and B are finitely axiomatized and sequential. We have:

$$A \rhd B \;\Leftrightarrow\; \mathsf{EA} \vdash \Diamond_{A,\rho(A)} \top \rightarrow \Diamond_{B,\rho(B)} \top.$$

Even better: $A \mapsto \mathsf{EA} + \Diamond_{A,\rho(A)} \top$ is an effective isomorphism between $\mathbb{D}_{\mathsf{seq}}$ and the Π_1-extensions of EA ordered by derivability.

It follows e.g. that the first-order theory of $\mathbb{D}_{\mathsf{seq}}$ is not arithmetical, by a result of Shavrukov in 2010.

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The Result

We show that, for finitely axiomatized sequential A, the structure $\mathbb{V}_A$ is convex in $\mathbb{D}_{\mathsf{seq}}$. This means that for very $B \rhd A$, there is a C in the same language as A with $C \vdash A$ with $C \equiv B$.

It follows that $\mathbb{V}_A$ inherits the suprema of $\mathbb{D}_{\mathsf{seq}}$.

We can also show that $\mathbb{V}_{\mathsf{Q}}$ is convex in $\mathbb{D}_{\mathsf{all}}$. Thus $\mathbb{V}_{\mathsf{Q}}$ inherits the suprema of $\mathbb{D}_{\mathsf{all}}$.

The suprema in $\mathbb{V}_{\mathsf{Q}}$ are in all but trivial cases different from the suprema of e.g. $\mathbb{V}_{\mathsf{PA}^-}$.

It follows from convexity that the first-order theory of $\mathbb{V}_A$ for consistent, finitely axiomatized sequential A is not arithmetical.

Švejdar's question remains open for $\mathbb{V}_A$ with $\mathsf{Q} \lhd A$ and A is not interderivable with Q and A is not mutually interpretable with a sequential theory.

The Proof 1

Consider a finitely axiomatized sequential A. Suppose $N : \mathsf{S}^1_2 \lhd A$. Let S be Σ_1. We find R such that:

$$\mathsf{S}^1_2 \vdash R \leftrightarrow S \leq \Box_{A,n} R^N.$$

Here n is 'large enough'.

This makes R an FGH-style fixed point (Friedman-Goldfarb-Harrington).

We can show:

$$\mathsf{EA} \vdash \Box_{A,n} R^N \leftrightarrow (S \vee \Box_{A,n} \bot).$$

Suppose $B \rhd A$. Taking $S : \Box_{B,\rho(B)} \bot$, we find:

$$\mathsf{EA} \vdash \Box_{A,n} R^N \leftrightarrow \Box_{B,\rho(B)} \bot.$$

The Proof 2

Take $Q := \neg R$. Contraposing:

$$\mathsf{EA} \vdash \Diamond_{A+Q^N,n}\top \leftrightarrow \Diamond_{B,\rho(B)}\top.$$

We find:

$$(A + Q^N) \equiv (\mathsf{S}^1_2 + \Diamond_{A+Q^N,n}\top) \equiv (\mathsf{S}^1_2 + \Diamond_{B,\rho(B)}\top) \equiv B.$$

So each $B \rhd A$ is mutually interpretable with a Π_1-extension relative to N of A. We can prove the same with Π_1 replaced with Σ_1.