Neologicism: for real(s)?

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Numbers and Truth

Outline

Slogan

Neologicism for real needs neologicism for reals.

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1. Target
2. Set theory

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Complete ordered fields: $\langle \mathbb{R}, +, \times, \leq \rangle$

$$\forall x, y \in \mathbb{R}\, \exists z \in \mathbb{R}\; z = x + y \qquad \text{(Closure)}$$

$$\forall x, y \in \mathbb{R}\, \exists z \in \mathbb{R}\; z = x \times y$$

$$\forall x, y, z \in \mathbb{R}\; x + (y + z) = (x + y) + z \qquad \text{(Associativity)}$$

$$\forall x, y, z \in \mathbb{R}\; x \times (y \times z) = (x \times y) \times z$$

$$\forall x, y \in \mathbb{R}\; x + y = y + x \qquad \text{(Commutativity)}$$

$$\forall x, y \in \mathbb{R}\; x \times y = y \times x$$

$$\exists y \in \mathbb{R}\, \forall x \in \mathbb{R}\; x + y = x \qquad \text{(Identity)}$$

$$\exists y \in \mathbb{R}\, \forall x \in \mathbb{R}\; x \times y = x$$

$$\forall x \in \mathbb{R}\, \exists y \in \mathbb{R}\; x + y = 0 \qquad \text{(Inverses)}$$

$$\forall x \in \mathbb{R}\, \exists y \in \mathbb{R}\; x \times y = 1$$

$$\forall x, y, z \in \mathbb{R}\; x \times (y + z) = (x \times y) + (x \times z) \qquad \text{(Distributivity)}$$

Complete ordered fields: $\langle \mathbb{R}, +, \times, \leq \rangle$

$\leq$: total order

$$\forall x, y, z \in \mathbb{R}\,[x \leq y \rightarrow x + z \leq y + z] \qquad \text{(Preservation)}$$

$$\forall x, y \in \mathbb{R}\,[0 \leq x \wedge 0 \leq y \rightarrow 0 \leq x \times y]$$

(Dedekind-complete)
Each non-empty subset of $\mathbb{R}$ with an ub in $\mathbb{R}$ has a lub.

Standard reduction

$$\langle x, y\rangle =_{df} \{\{x\}, \{x, y\}\} \qquad \text{(Pairsdf)}$$

$$\langle x, y\rangle = \langle x', y'\rangle \equiv x = x' \wedge y = y' \qquad \text{(Pairs)}$$

$$\begin{aligned} 0 &= \emptyset \qquad \text{(Natural)} \\ 1 &= \{0\} = \{\emptyset\} \\ 2 &= \{0, 1\} = \{\emptyset, \{\emptyset\}\} \\ 3 &= \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \\ &\vdots \end{aligned}$$

Standard reduction

$$\text{Eqivalence classes on pairs } m, n{:}m - n, m, n \in \mathbb{N} \qquad \text{(Integers)}$$

$$x - y = x' - y' \equiv x + y' = x' + y \qquad \text{(Difference)}$$

$$\text{Eqivalence classes on pairs } x, y, y \neq 0 \qquad \text{(Rationals)}$$

$$\frac{x}{y} = \frac{x'}{y'} \equiv x \times y' = x' \times y \qquad \text{(Ratio I)}$$

(Dedekind cuts)
$\langle A, B\rangle$ of sets of rational numbers, $A, B \neq \emptyset$, A is closed downwards, B is closed upwards, A contains no greatest element

Standard reduction

Problem

The embarrassment of riches (Benacerraf-style).

$$\{\{\emptyset\}\} =_? 2 =_? \{\emptyset, \{\emptyset\}\}$$

Piece of Albert Visser's wisdom, a few hours ago

"As a sui-genericist, I am skeptical about this one."

Neologicism: the main gist

APs: the notion

$$\underbrace{\Sigma(\alpha)}_{\text{singular term}} = \Sigma(\beta) \equiv \underbrace{\alpha}_{\text{singular t. or predicate}} \overbrace{\sim}^{\text{equivalence rel.}} \beta$$

Hume's Principle: PA, consistency, analyticity(?)

$$N(F) = N(G) \equiv F \sim G \qquad \text{(Hume)}$$

General idea

Build mathematical theories using APs.

Neologicism: the main gist

Extensions, BLV, inconsistency

$$\{x \mid Fx\} = \{x \mid Gx\} \equiv \forall x\,(Fx \equiv Gx) \qquad \text{(BLV)}$$

Problem: cardinality issues

Challenge: acceptability criteria?

Neologicism: the main gist

Fun fact about neologicism

No embarrassment of riches!
(objects sui generis, equivalent right-hand side conditions)

Not so fun fact about neologicism

Not too many hills conquered! What about, say, real numbers?

Simons (1987) on real numbers
(contra (Hale and Wright, 2001))

- Reals vs. magnitudes (one vs. many).
- Mangitudes as pairs: natural numbers and bicimals

$$a = n + \sum_{k=1}^{\infty}[\frac{1}{2^{n_k}}]$$

- Addition: quantify over lengths of truncates.
- To get positive reals (Euclid, sort of):

$$\frac{a}{b} = \frac{c}{d} \equiv \forall n, m \in \mathbb{N}\,[a^n \leq / = / \geq b^m \text{ as } c^n \leq / = / \geq d^m] \quad \text{(Ratio II)}$$

- Negative reals: informally.

Simons (1987) on real numbers
(contra (Hale and Wright, 2001))

Minor issues

- Motivation for magnitudes vs. reals fails.
- Level of ratios seems redundant.

$$R(F) = R(G) \equiv \forall x \in \mathbb{N}\, (Fx \equiv Gx) \qquad \text{(Brute-force)}$$

(No sets! Somehow define operations in a kosher manner!)

A more serious problems:

Operations and their properties not clearly within theory.

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A really serious problem (**or not**): Frege's constraint

A satisfactory foundation for any branch of mathematics should somehow also explain its basic concepts so that their applications are immediate (Wright, 2000).

Shapiro's reals

$$x - y = x' - y' \equiv x + y' = x' + y \qquad \text{(Integers)}$$

$$Q(m, n) = Q(p, q) \equiv (n = 0 \wedge q = 0) \vee \qquad \text{(Quotients)}$$
$$\vee\, (n \neq 0 \wedge q \neq 0 \wedge m \times q = n \times p)$$

Rationals: $Q(m, n)$ with $n \neq 0$.

$$P \leq r \equiv_{df} \forall x\, (Px \rightarrow x \leq r)$$

Say P of rationals is a cut-like property if it is bounded above and instantiated.

$$C(P) = C(Q) \equiv \forall r\, [P \leq r \equiv Q \leq r] \qquad \text{(Cut)}$$

Real numbers: cuts of cut-like properties.

Shapiro's operations: staying real

$$(a-b)+(c-d)=(a+c)-(b+d) \tag{1}$$
$$(a-b)\times(c-d)=(a\times c+b\times d)-(b\times c+a\times d) \tag{2}$$
$$Q(m,n)+Q(p,q)=Q(m\times q+p\times n, n\times q) \tag{3}$$
$$Q(m,n)\times Q(p,q)=Q(m\times p, n\times q) \tag{4}$$

r rational; P, Q cut-like. Define:

$$C(P) < C(Q) \equiv C(P) \neq C(Q) \wedge \forall r\,(Q \leq r \rightarrow P \leq r)$$

r is a ZERO rational number iff $r < 0$.
Real 0 (additive identity) is $C(ZERO)$.
The additive inverse of $C(P)$ is $C(-P)$, where $-Pr$ iff $P \leq -r$.
r is a $P+Q$-rational number iff $\exists x, y\,(Px \wedge Qy \wedge r < x+y)$.
Define:

$$C(P)+C(Q)=C(P+Q)$$

Shapiro: philosophical concerns

Dissimilarity between Hume and reals

"...it's clear that individual quantities don't have their real numbers after the fashion in which a particular concept ... has its cardinal number. We are familiar with different systems of measurement, like the imperial and metric systems for lengths, volumes, and weights ... but there is no conceptual space for correspondingly different systems of counting. Of course, there can be different systems of counting notation: we can count in a decimal or binary system, for instance, or in Roman or Arabic numerals. But if they are used correctly, they wont differ in the cardinal number they deliver to any specied concept, but only in the way they name that number. By contrast, the imperial and metric systems do precisely differ in the real numbers they assign to the length of a specied object. The real number properly assigned to a length depends on a previously fixed unit of comparison." (Hale and Wright, 2005, 190), (Wright, 2000, 5)

Shapiro: philosophical concerns

Frege's constraint once again

(Hume): second order properties of concepts.
(Cut): properties of cut-like properties of rational numbers!?

This goes against

- Intuitions that reals are properties of magnitudes or objects relative to a certain way of measuring them.
- Hempel (1952, 63), who takes quantitative concepts to be functions assigning real numbers to objects from a given domain, and
- Carnap (1966, 62), who says that measurement is an assignment of numbers to a body or process.

Shapiro: philosophical concerns

Biting the bullet

- Staying structuralist?
- Separating axiomatization from explanation (Wright)?

Hale's quantitative domains

- Things in q. relations vs. quantities introduced by APs. *the length of a = the length of b iff a is as long as b*
- Mere availability of "more ϕ than" or "as ϕ as" doesn't make ϕ a quantity (e.g. elegant, emotionally intelligent).
- What does? Some combination operation which preserves ordering: $a \oplus b >_\phi a, b$

Hale's quantitative domains

minimal Q-domain

A set Q with operation $\oplus$ which commutes, associates and satisfies the trichotomy condition, according to which exactly one of $\exists c\, a = b \oplus c$, $\exists c\, b = a \oplus c$ and $a = b$ holds (order $a < b$ is then defined by the second of these).

Hale's quantitative domains

To exclude infinite and infinitely small quantities Hale moves to:

normal Q-domains

Minimal Q-domains + the comparability condition:

$$\forall a, b\, \exists m\, (ma > b), m \in \mathtt{N}^{+}$$

For normal Q, Q', (Ratio III):

$$\forall a, b \in Q, c, d \in Q'\; a : b = c : d \equiv$$
$$\equiv \forall n, m \in \mathtt{N}\, [ma \leq / = / \geq nb \text{ as } mc \leq / = / \geq nd]$$

Hale's quantitative domains

To ensure common denominators and no smallest quantity:

Full Q-domains

A normal Q-domain for which $\forall a, b, c \in Q\, \exists q \in Q\; a : b = q : c$.

Hale's quantitative domains

To ensure uncountability:

Complete Q-domains

A full Q-domain where every bounded above and non-empty set $S \subseteq Q$ has a lub.

Real numbers

Ratios over complete Q-domains: positive reals. To obtain negative reals use (Difference).

Hale's quantitative domains

The challenge

Prove the existence of complete Q-domains.
"There is no prior guarantee that the physical world comprises real-valued quantities."
Hale's way out: build over natural numbers (cut).

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Weak spots

- Frege's constraint: better but not perfect.
- The applicability of real numbers to a domain depends on its being a complete Q-domain (and this would mean that we *don't know* if real numbers are useful when it comes to handling real objects).
- The existence proof of a complete Q-domain still depends on a structuralist cut construction.

HP and reals: analogy?

A seemingly unrelated question...

Is being a father a property of a person? Yes, but relational.

HP and reals: analogy?

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...and its relevance

Is there being n frogs in Sweden a property of the concept? Yes, but relational.

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Is there being *n* frogs in Sweden a property of the concept? Yes, but relational.

Hidden argument of (Hume)

"one and the same physical entity might be conceptualized as consisting of 1 army, 5 divisions, 20 regiments, 100 companies, etc" (Zalta 2008)

$$N(Stuff, P) = N(Stuff, Q) \equiv P \sim Q$$

Taking a leap

Actual quantities

Stuff + predicate = actual domain

actual domain + AP = actual quantities

weight(StuffInSweden,Frog,Fet)=weight(StuffInSweden,Frog,Hals) iff **E**(Fet,Hals) (usually extralogical, perhaps operational)

Taking a leap

Potential abstract operations

- Composition (commutes, associates, trichotomy(?))

$$[w(Fet) \oplus w(Hals)] = [w(Fet') \oplus w(Hals')] \equiv$$

(modal?) $$\equiv \mathbf{E}^{\oplus}(Fet, Hals, Fet', Hals')?$$

- Multiplication by recursive clauses.
- Define order, preserved by $\oplus$: comparability (no infinitely heavy/light frogs)
- Ratios by abstraction.
- Fullness and completeness?
- Difference by abstraction.

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Reals numbers

Pick a unit u. Then: $R_u(a) = a : u$.

Summary

- Simons: Not within the system, not sure if kosher.
- Shapiro: within the system, but purely structural
- Hale: related to real world, some existence and applicability issues
- Current proposal: if there are frogs in Sweden, we have real numbers. Handles Hale's issues.

Summary

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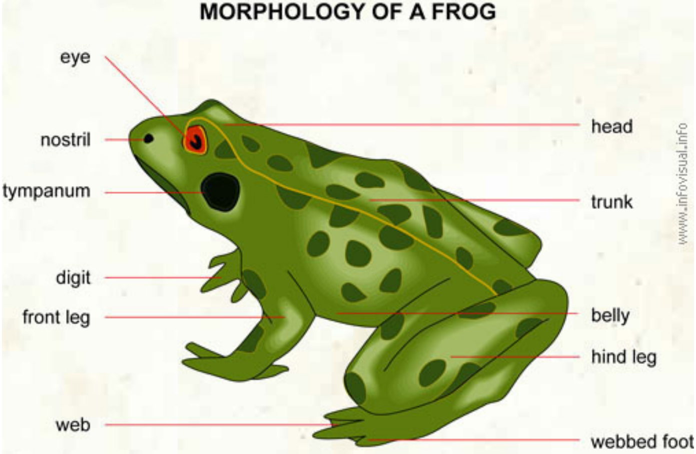
The price

- Justify the neologicist acceptability of the operations involved!
- Argue that those operations have the right properties!
- Modalities seem to be brought in!
- Mathematicians won't care!

Thank you!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eye
nostril
tympanum
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