The Disentanglement of Syntax from a model theoretic point of view

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Tarski into axioms: CT-

- Tarski's definition of truth can be turned into an axiomatic version with the theory CT- (for Compositional Truth)
- CT- is the theory, in the language

  Ltr := Lpa ∪ {Tr}

  consisting of the usual axioms of PA in Lpa (no full induction) plus the truth theoretic axioms:

Truth axioms of CT-

1. $\forall x(\text{Atomic-pa}(x) \rightarrow (Tr(x) \leftrightarrow Tr^*(x)))$;

2. $\forall x(\text{Sent-pa}(x) \rightarrow (Tr(neg(x)) \leftrightarrow \neg Tr(x)))$;

3. $\forall x \forall y(\text{Sent-pa}(x) \& \text{Sent-pa}(x) \rightarrow (Tr(Conj(x,y)) \leftrightarrow Tr(x) \& Tr(y)))$

4. $\forall x \forall z(\text{Formpa}(x) \& Var(z) \rightarrow (Tr(Un(x,z)) \leftrightarrow \forall y Tr(Sub(x,y,\mathbf{y},))))$

Syntax included

- In order to have meaningful axioms for truth, some syntactic information is needed.

  (We employed notions like “Sent-pa(x)” or “Conj(x)”)

- A theory of syntax must be included in CT-. This is why we added truth axioms to PA. In PA we can develop syntax for Lpa, in the well known manner.

Syntax for free

- Why using PA as syntax, instead of, to say, a concatenation theory?

  - We usually want to investigate what happens if axioms for truth are added to an arithmetical theory like PA, so we can avoid the annoying addition of an independent syntax theory.

- If PA is our object theory, we gain syntax for free.

Non standard sentences

- Accordingly, in PA we can define a formula "Sent-pa(x)" such that N $\models$ Sent-pa(n) is true if and only if n is a code of a sentence of Lpa. As expected.
- But if we move from N to a non standard model M, we also get that M $\models$ Sent-pa(b) for some b $\in$ M, and b non standard. (because of the Overspill Principle). We must admit non standard sentences.

...and non standard truths

- Each axiom of CT- is introduced by a clause stating that “Tr(x)” applies to any element satisfying “Sent-pa(x)”.
- Thus, some non standard element *can* enter the extension of the truth predicate, when M is non standard.
- Some non standard element b *must* enter it, actually, since

  CT- $\vdash \forall x\{\text{Sent-pa}(x)\rightarrow[(\text{Tr}(x) \vee \text{Tr}(\text{neg}(x))]\}$

  so if M $\models$ Sent-pa(b) either (the sentence coded by) b or the negation of b will enter the extension of Tr(x)

Satisfaction Classes

- if M, S $\models$ CT-, and S is a set giving a suitable extension for "Tr(x)" in M,

  then we say that S is a Satisfaction Class for M.
- This explains why CT- is also called PA + "there is a (full) satisfaction class".
- The basic result about CT- is that it is conservative over PA but...

Satisfaction Classes

...conservativity of CT- should be contrasted with the fact that not every model of PA has a Satisfaction Class.

Lachlan's theorem: *if M is a non standard model of PA admitting a (full) satisfaction class S, then M is recursively saturated.*

- Thus, not in every M ⊨ PA there is a set S satisfying the truth predicate of CT-.
- This is already remarkable, but, moreover, apparently, we do have a suitable extension in any model...

...and Th(M)?

- Take a non standard model M. Now consider Th(M), namely, the set of standard sentences true in M.

  We clearly have Th(M) in every model of PA.

- Now consider the set of elements of M coding the sentences in Th(M). Call this set T.
- Here we go! T is a suitable extension for "Tr(x)", isn't it?

...and Th(M)?

- Th(M) is defined by "M $\models$ ", which is based on Tarski's definition of truth.

  Thus Th(M) satisfy compositional clauses, and so T should be a good interpretation for the truth predicate of CT-, whatever model M we pick.

- The problem is that such an argument is wrong.

...and Th(M)?

- The obstacle is that Th(M) is a set of *standard* sentences, but CT- forces *non standard* sentences to enter the extension of Tr(x).
- Once non standard sentences are in play, the compositional axioms can be applied to them, and we can make Lachlan's proof work.
- Non standard sentences are crucial in the proof of Lachlan's theorem.

Getting rid of non standard sentences

- If we could get rid of non standard sentences, we might be able to recover our intuitive argument and give the intuitive extension to Tr“x”.
- Non standard sentences are problematic even from a general point of view: we do not recognize them as actual sentences. So, we shouldn't be forced to embrace non standard sentences, and apply the truth predicate to them, only because the object theory has non standard models.

Disentagling syntax from the object theory

- One of the basic reasons why we always have to do with non standard sentences in CT- is not hard to identify: CT- is formulated by quantifying through the formula of the *object* language "Sent-pa(x)".
- hence, we face non standard sentences anytime the model of the object theory is non standard.
- This is an immediate consequence of the fact that we developed the syntax theory inside the object theory, we had better separate them.

Disentagling syntax from the object theory

- Here is a sketchy hint of how this could be done:

  We need:

  - 1. a theory of syntax S in the language Ls;

  - 2. a theory of truth TS for Lpa yielded by adding truth axioms in Lt to S (and not to PA);

  - 3. a (separate) object theory, like PA in Lpa.

Disentagling syntax from the object theory

- Any model of PA can be expanded to a model of TS ∪ PA (no induction extended).
- The proof will depend on the details of the construction, but the relevant move is that now we can always interpret syntax in the standard fragment of each model M of PA.
- The construction is basically the same used by Craig and Vaught to get finite axiomatizability of theories (with infinite models only).

Conclusion

- The disentaglement of syntax has been recently (re)-proposed by Richard Heck, and critically reviewed by Volker Halbach.
- Their reflections moved mainly from proof-theoretical results. Here I tried to give a different motivation for the general enterprise from another, more model theoretic point of view.
- But the bulk of the work is still to be done.