Model Theory of Satisfaction Classes

Roman Kossak
CUNY

Numbers and Truth
Gothenburg, October 2012

50 years of nonstandard satisfaction

1. A. Robinson, *On languages based on non-standard arithmetic,* Nagoya Mathematical Journal, 1963.
2. S. Krajewski, *Nonstandard satisfaction classes,* Springer Lecture Notes in Mathematics, 537, 1976.
3. H. Kotlarski, S. Krajewski, and A. Lachlan, *Construction of satisfaction classes for non-standard models,* Canadian Mathematical Bulletin, 1981
4. A. Lachlan, *Full satisfaction classes and recursive saturation,* Canadian Mathematical Bulletin, 1981.
5. ... F. Engström, H. Kotlarski, R. Murawski, Z. Ratajczyk, J. Schmerl, S. Smith, ...
6. A. Enayat, A. Visser, *Full satisfaction classes in a general setting, part 1* to appear.

50 years of nonstandard satisfaction

1. A. Robinson, *On languages based on non-standard arithmetic,* Nagoya Mathematical Journal, 1963.
2. S. Krajewski, *Nonstandard satisfaction classes,* Springer Lecture Notes in Mathematics, 537, 1976.
3. H. Kotlarski, S. Krajewski, and A. Lachlan, *Construction of satisfaction classes for non-standard models,* Canadian Mathematical Bulletin, 1981
4. A. Lachlan, *Full satisfaction classes and recursive saturation,* Canadian Mathematical Bulletin, 1981.
5. ... F. Engström, H. Kotlarski, R. Murawski, Z. Ratajczyk, J. Schmerl, S. Smith, ...
6. A. Enayat, A. Visser, *Full satisfaction classes in a general setting, part 1* to appear.

50 years of nonstandard satisfaction

1. A. Robinson, *On languages based on non-standard arithmetic,* Nagoya Mathematical Journal, 1963.
2. S. Krajewski, *Nonstandard satisfaction classes,* Springer Lecture Notes in Mathematics, 537, 1976.
3. H. Kotlarski, S. Krajewski, and A. Lachlan, *Construction of satisfaction classes for non-standard models,* Canadian Mathematical Bulletin, 1981
4. A. Lachlan, *Full satisfaction classes and recursive saturation,* Canadian Mathematical Bulletin, 1981.
5. ... F. Engström, H. Kotlarski, R. Murawski, Z. Ratajczyk, J. Schmerl, S. Smith, ...
6. A. Enayat, A. Visser, *Full satisfaction classes in a general setting, part 1* to appear.

Robinson's motivation: semantics for infinitary languages

$M \models \mathrm{PA}$, $c > \omega$.

$$\varphi_1 \wedge \varphi_2 \wedge \cdots$$

$$(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_c) \in M$$

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \ldots \varphi(x_1, x_2 \; \ldots)$$

$$(\forall x_1 \exists y_1, \forall x_2 \exists y_2 \ldots \forall x_c \exists y_c \varphi(x_1, x_2, \ldots)) \in M$$

Robinson's motivation: semantics for infinitary languages

$M \models \mathrm{PA}$, $c > \omega$.

$$\varphi_1 \wedge \varphi_2 \wedge \cdots$$

$$(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_c) \in M$$

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \ldots \varphi(x_1, x_2 \ldots)$$

$$(\forall x_1 \exists y_1, \forall x_2 \exists y_2 \ldots \forall x_c \exists y_c \varphi(x_1, x_2, \ldots)) \in M$$

Robinson's motivation: semantics for infinitary languages

$M \models \mathrm{PA}$, $c > \omega$.

$$\varphi_1 \wedge \varphi_2 \wedge \cdots$$

$$(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_c) \in M$$

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \ldots \varphi(x_1, x_2 \ \ldots)$$

$$(\forall x_1 \exists y_1, \forall x_2 \exists y_2 \ldots \forall x_c \exists y_c \varphi(x_1, x_2, \ldots)) \in M$$

Robinson's motivation: semantics for infinitary languages

$M \models \mathrm{PA}$, $c > \omega$.

$$\varphi_1 \wedge \varphi_2 \wedge \cdots$$

$$(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_c) \in M$$

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \ldots \varphi(x_1, x_2 \ \ldots)$$

$$(\forall x_1 \exists y_1, \forall x_2 \exists y_2 \ldots \forall x_c \exists y_c \varphi(x_1, x_2, \ldots)) \in M$$

My motivation: model theory of countable recursively saturated models of PA

Model theory of countable recursively saturated models of PA $=$ *Model theory of countable models* (M, S), *where* S *is a partial inductive satisfaction class for* M.

Model theory of countable recursively saturated models of PA $=$ *results about* $\mathrm{Lt}(M)$, $\mathrm{Aut}(M)$ *and* $\mathrm{Cut}(M)$.

My motivation: model theory of countable recursively saturated models of PA

Model theory of countable recursively saturated models of PA $=$ *Model theory of countable models* (M, S), *where* S *is a partial inductive satisfaction class for* M.

Model theory of countable recursively saturated models of PA $=$ *results about* $\mathrm{Lt}(M)$, $\mathrm{Aut}(M)$ *and* $\mathrm{Cut}(M)$.

Truth extensions

Definition

$S \subseteq M \models \mathrm{PA}$ *is a* *truth extension* *is for all* $\varphi \in \mathcal{L}_{\mathrm{PA}}(M)$,

$$\ulcorner \varphi \urcorner \in S \Leftrightarrow M \models \varphi.$$

Proposition (Tarski)

No truth extension is definable.

Proposition

Let M *be countable nonstandard model of* PA. *T.f.a.e.*

1. M *is* *recursively saturated*;
2. M *has an* *inductive* *truth extension, i.e. a truth extension* S *such that* $(M, S) \models \mathrm{PA}^*$;
3. M *has a truth extension* S *such that* $(M, S) \models I\Sigma_1$.

Truth extensions

Definition

$S \subseteq M \models \mathrm{PA}$ *is a* *truth extension* *is for all* $\varphi \in \mathcal{L}_{\mathrm{PA}}(M)$,

$$\ulcorner\varphi\urcorner \in S \Leftrightarrow M \models \varphi.$$

Proposition (Tarski)

No truth extension is definable.

Proposition

Let M be countable nonstandard model of PA. *T.f.a.e.*

1. *M is recursively saturated;*
2. *M has an inductive truth extension, i.e. a truth extension S such that $(M, S) \models \mathrm{PA}^*$;*
3. *M has a truth extension S such that $(M, S) \models I\Sigma_1$.*

Truth extensions

Definition

$S \subseteq M \models \mathrm{PA}$ *is a* *truth extension* *is for all* $\varphi \in \mathcal{L}_{\mathrm{PA}}(M)$,

$$\ulcorner \varphi \urcorner \in S \Leftrightarrow M \models \varphi.$$

Proposition (Tarski)

No truth extension is definable.

Proposition

Let M *be countable nonstandard model of* PA. *T.f.a.e.*

1. M *is* *recursively saturated*;
2. M *has an* *inductive* *truth extension, i.e. a truth extension* S *such that* $(M, S) \models \mathrm{PA}^*$;
3. M *has a truth extension* S *such that* $(M, S) \models I\Sigma_1$.

Truth extensions

Definition

$S \subseteq M \models \mathrm{PA}$ is a truth extension is for all $\varphi \in \mathcal{L}_{\mathrm{PA}}(M)$,

$$\ulcorner \varphi \urcorner \in S \Leftrightarrow M \models \varphi.$$

Proposition (Tarski)

No truth extension is definable.

Proposition

Let M be countable nonstandard model of PA. T.f.a.e.

1. *M is recursively saturated;*
2. *M has an inductive truth extension, i.e. a truth extension S such that $(M, S) \models \mathrm{PA}^*$;*
3. *M has a truth extension S such that $(M, S) \models I\Sigma_1$.*

From inductive truth extension to saturation

Let $p(x, a)$ be a recursive type with $a \in M$, let $P(v, w) \in \Sigma_1$ be such that

$$\{i \in \omega : M \models P(i, a)\} = \{\ulcorner \varphi(x, y) \urcorner : \varphi(x, a) \in p(x, a)\}.$$

Then for $n \in \omega$

$$M \models \exists x \forall i < n[P(i, a) \Rightarrow i(x, a) \in S]$$

Hence, for come nonstandard c

$$M \models \exists x \forall i < c[P(i, a) \Rightarrow i(x, a) \in S]$$

From inductive truth extension to saturation

Let $p(x, a)$ be a recursive type with $a \in M$, let $P(v, w) \in \Sigma_1$ be such that

$$\{i \in \omega : M \models P(i, a)\} = \{\ulcorner \varphi(x, y) \urcorner : \varphi(x, a) \in p(x, a)\}.$$

Then for $n \in \omega$

$$M \models \exists x \forall i < n [P(i, a) \Rightarrow i(x, a) \in S]$$

Hence, for come nonstandard c

$$M \models \exists x \forall i < c [P(i, a) \Rightarrow i(x, a) \in S]$$

From inductive truth extension to saturation

Let $p(x,a)$ be a recursive type with $a \in M$, let $P(v,w) \in \Sigma_1$ be such that

$$\{i \in \omega : M \models P(i,a)\} = \{\ulcorner \varphi(x,y) \urcorner : \varphi(x,a) \in p(x,a)\}.$$

Then for $n \in \omega$

$$M \models \exists x \forall i < n[P(i,a) \Rightarrow i(x,a) \in S]$$

Hence, for come nonstandard c

$$M \models \exists x \forall i < c[P(i,a) \Rightarrow i(x,a) \in S]$$

From resplendence to inductive truth extensions

Let M be a resplendent model and let T be the recursive theory in $\mathcal{L}_{\mathrm{PA}} \cup \{S\}$ consisting of

1. $\mathrm{PA}(S)$;
2. $\{\forall x[\varphi(x) \in S \Leftrightarrow \varphi(x)] : \varphi(x) \in \mathcal{L}_{\mathrm{PA}}]\}$.

Every finite fragment of T has a model of the form (M, X), where $X \in \mathrm{Def}(M)$, hence there is $S \subseteq M$ such that $(M, S) \models T$.

Moreover, if M is countable, the there is and S such that $(M, S) \models T$ and (M, S) is resplendent.

From resplendence to inductive truth extensions

Let M be a resplendent model and let T be the recursive theory in $\mathcal{L}_{\mathrm{PA}} \cup \{S\}$ consisting of

1. $\mathrm{PA}(S)$;
2. $\{\forall x[\varphi(x) \in S \Leftrightarrow \varphi(x)] : \varphi(x) \in \mathcal{L}_{\mathrm{PA}}]\}$.

Every finite fragment of T has a model of the form (M, X), where $X \in \mathrm{Def}(M)$, hence there is $S \subseteq M$ such that $(M, S) \models T$.

Moreover, if M is countable, the there is and S such that $(M, S) \models T$ and (M, S) is resplendent.

From resplendence to inductive truth extensions

Let M be a resplendent model and let T be the recursive theory in $\mathcal{L}_{\mathrm{PA}} \cup \{S\}$ consisting of

1. $\mathrm{PA}(S)$;
2. $\{\forall x[\varphi(x) \in S \Leftrightarrow \varphi(x)] : \varphi(x) \in \mathcal{L}_{\mathrm{PA}}]\}$.

Every finite fragment of T has a model of the form (M, X), where $X \in \mathrm{Def}(M)$, hence there is $S \subseteq M$ such that $(M, S) \models T$.

Moreover, if M is countable, the there is and S such that $(M, S) \models T$ and (M, S) is resplendent.

Tarski's conditions

Let $Q_n = \Sigma_0(\Sigma_n \cup \Pi_n)$.

Proposition

If $M \models \mathrm{PA}$ is nonstandard and S is an inductive truth extension for M then there is a $e > \omega$ such that for all $\varphi, \psi \in Q_e(M)$

1. $(\varphi \wedge \psi) \in S$ *iff* $\varphi \in S$ *and* $\psi \in S$;
2. $\varphi \in S$ *iff* $\neg\varphi \notin S$;
3. *If* $\exists x \varphi(x) \in Q_e$, *then* $\exists x \varphi(x) \in S$ *iff* $\varphi(b) \in S$, *for some* $b \in M$.

Definition

If a truth extension S satisfies (1), (2) and (3) above it is e-full.
If S is e-full for all $e \in M$, it is full.
If S is e-full, then $S \cap Q_e(M)$ is a partial satisfaction class for M.

Tarski's conditions

Let $Q_n = \Sigma_0(\Sigma_n \cup \Pi_n)$.

Proposition

If $M \models \mathrm{PA}$ is nonstandard and S is an inductive truth extension for M then there is a $e > \omega$ such that for all $\varphi, \psi \in Q_e(M)$

1. *$(\varphi \wedge \psi) \in S$ iff $\varphi \in S$ and $\psi \in S$;*
2. *$\varphi \in S$ iff $\neg\varphi \notin S$;*
3. *If $\exists x \varphi(x) \in Q_e$, then $\exists x \varphi(x) \in S$ iff $\varphi(b) \in S$, for some $b \in M$.*

Definition

If a truth extension S satisfies (1), (2) and (3) above it is e-full.
If S is e-full for all $e \in M$, it is full.
If S is e-full, then $S \cap Q_e(M)$ is a partial satisfaction class for M.

Full inductive satisfaction classes

Example

Let $\mathbb{N} = (\omega, +, \times)$ *and let* $S_{\mathbb{N}} = \{\ulcorner\varphi\urcorner : \mathbb{N} \models \varphi\}$. *Then* $S_{\mathbb{N}}$ *is a full inductive satisfaction class for* $\mathbb{N}$*, and if* $(\mathbb{N}, S_{\mathbb{N}}) \prec (M, S)$*, then* S *is a full inductive satisfaction class for* M*.*

Example

$T = \{\varphi \in \mathcal{L}_{\mathrm{PA}} : \mathrm{PA} + S \text{ is a full inductive satisfaction class} \vdash \varphi\}$.
If $M \models T + \neg\mathrm{Con}(T)$*, then* $M \not\models \mathrm{TA}$ *and, if* M *is resplendent, then it has a full inductive satisfaction class.*

Full inductive satisfaction classes

Example

Let $\mathbb{N} = (\omega, +, \times)$ *and let* $S_{\mathbb{N}} = \{\ulcorner\varphi\urcorner : \mathbb{N} \models \varphi\}$. *Then* $S_{\mathbb{N}}$ *is a full inductive satisfaction class for* $\mathbb{N}$, *and if* $(\mathbb{N}, S_{\mathbb{N}}) \prec (M, S)$, *then* S *is a full inductive satisfaction class for* M.

Example

$T = \{\varphi \in \mathcal{L}_{\mathsf{PA}} : \mathsf{PA} + S \text{ is a full inductive satisfaction class} \vdash \varphi\}$.
If $M \models T + \neg\mathrm{Con}(T)$, *then* $M \not\models \mathsf{TA}$ *and, if* M *is resplendent, then it has a full inductive satisfaction class.*

Kaufmann model

Definition

1. *For* $X \subseteq M \models \mathrm{PA}$, $X \in \mathrm{Class}(M)$ *iff* $\forall a \in M\ (X \cap \{0, 1, \dots, a\}) \in \mathrm{Def}(M)$.
2. M *is rather classless if* $\mathrm{Class}(M) = \mathrm{Def}(M)$.

Theorem (Kaufmann+$\Diamond$, Shelah in ZFC)

There are rather classless recursively saturated models of PA.

Corollary

There are recursively saturated models without partial inductive satisfaction classes, and (S. Smith) without full satisfaction classes.

Theorem (Schmerl)

There are recursively saturated models without partial inductive satisfaction classes that are not rather classless.

Kaufmann model

Definition

1. *For* $X \subseteq M \models \mathrm{PA}$, $X \in$ Class(M) *iff* $\forall a \in M \ (X \cap \{0, 1, \dots, a\}) \in \mathrm{Def}(M)$.
2. M *is* *rather classless* *if* $\mathrm{Class}(M) = \mathrm{Def}(M)$.

Theorem (Kaufmann+$\Diamond$, Shelah in ZFC)

There are rather classless recursively saturated models of PA.

Corollary

There are recursively saturated models without partial inductive satisfaction classes, and (S. Smith) without full satisfaction classes.

Theorem (Schmerl)

There are recursively saturated models without partial inductive satisfaction classes that are not rather classless.

Kaufmann model

Definition

1. *For* $X \subseteq M \models \mathrm{PA}$, $X \in \mathrm{Class}(M)$ *iff* $\forall a \in M\ (X \cap \{0, 1, \ldots, a\}) \in \mathrm{Def}(M)$.
2. M *is rather classless if* $\mathrm{Class}(M) = \mathrm{Def}(M)$.

Theorem (Kaufmann+$\Diamond$, Shelah in ZFC)

There are rather classless recursively saturated models of PA.

Corollary

There are recursively saturated models without partial inductive satisfaction classes, and (S. Smith) without full satisfaction classes.

Theorem (Schmerl)

There are recursively saturated models without partial inductive satisfaction classes that are not rather classless.

Kaufmann model

Definition

1. *For* $X \subseteq M \models \mathrm{PA}$, $X \in \mathrm{Class}(M)$ *iff* $\forall a \in M \ (X \cap \{0, 1, \ldots, a\}) \in \mathrm{Def}(M)$.
2. M *is rather classless if* $\mathrm{Class}(M) = \mathrm{Def}(M)$.

Theorem (Kaufmann+$\Diamond$, Shelah in ZFC)

There are rather classless recursively saturated models of PA.

Corollary

There are recursively saturated models without partial inductive satisfaction classes, and (S. Smith) without full satisfaction classes.

Theorem (Schmerl)

There are recursively saturated models without partial inductive satisfaction classes that are not rather classless.

Partial satisfaction classes

Proposition (Krajewski)

If M has a full inductive satisfaction class, then $M \models Con(\mathrm{PA})$ (and much more by results of Kotlarski and Ratajczyk).

Proposition

1. *If S is an inductive satisfaction class for a nonstandard M, and S is not full, then there is a maximum $e > \omega$ such that S is e-full.*
2. *If S an e-full inductive satisfaction class for M, then for each $n \in \omega$ there is an $(e+n)$-full inductive satisfaction class $S_n \in \mathrm{Def}(M, S)$.*
3. *(Kotlarski) If S is an e-full inductive satisfaction class for M and for all $n \in \omega$, $d + n < e$, then $(M, S \cap Q_d(M))$ is recursively saturated.*

Partial satisfaction classes

Proposition (Krajewski)

If M has a full inductive satisfaction class, then $M \models Con(\mathrm{PA})$ (and much more by results of Kotlarski and Ratajczyk).

Proposition

1. *If S is an inductive satisfaction class for a nonstandard M, and S is not full, then there is a maximum $e > \omega$ such that S is e-full.*
2. *If S an e-full inductive satisfaction class for M, then for each $n \in \omega$ there is an $(e + n)$-full inductive satisfaction class $S_n \in \mathrm{Def}(M, S)$.*
3. *(Kotlarski) If S is an e-full inductive satisfaction class for M and for all $n \in \omega$, $d + n < e$, then $(M, S \cap Q_d(M))$ is recursively saturated.*

Partial satisfaction classes

Proposition (Krajewski)

If M has a full inductive satisfaction class, then $M \models Con(\mathrm{PA})$ (and much more by results of Kotlarski and Ratajczyk).

Proposition

1. *If S is an inductive satisfaction class for a nonstandard M, and S is not full, then there is a maximum $e > \omega$ such that S is e-full.*
2. *If S an e-full inductive satisfaction class for M, then for each $n \in \omega$ there is an $(e+n)$-full inductive satisfaction class $S_n \in \mathrm{Def}(M, S)$.*
3. *(Kotlarski) If S is an e-full inductive satisfaction class for M and for all $n \in \omega$, $d + n < e$, then $(M, S \cap Q_d(M))$ is recursively saturated.*

Partial satisfaction classes

Proposition (Krajewski)

If M has a full inductive satisfaction class, then $M \models Con(\mathrm{PA})$ (and much more by results of Kotlarski and Ratajczyk).

Proposition

1. *If S is an inductive satisfaction class for a nonstandard M, and S is not full, then there is a maximum $e > \omega$ such that S is e-full.*
2. *If S an e-full inductive satisfaction class for M, then for each $n \in \omega$ there is an $(e+n)$-full inductive satisfaction class $S_n \in \mathrm{Def}(M, S)$.*
3. *(Kotlarski) If S is an e-full inductive satisfaction class for M and for all $n \in \omega$, $d + n < e$, then $(M, S \cap Q_d(M))$ is recursively saturated.*

Proof:

Lemma

If S is an e-full inductive satisfaction class for M, then for all $d < e$

$$(M, S) \models \forall \varphi \in Q_d(M)[S(\varphi) \Leftrightarrow S(\mathrm{Tr}_d(\varphi))].$$

Proof.

By induction on d. □

Let $S_d = S \cap Q_d(M)$ and let $\Phi(S_d)$ be a (standard) sentence of $(\mathcal{L}_{\mathrm{PA}} \cup \{S_d\})(M)$. Then there are $n \in \omega$ and $\Phi^* \in Q_{d+n}(M)$ such that

$$(M, S_d) \models \Phi \Leftrightarrow (M, S) \models S(\Phi^*).$$

Define $(S_d(x))^* = \mathrm{Tr}_d(x)$ and then define Φ^* by induction on complexity of Φ.

Proof:

Lemma

If S is an e-full inductive satisfaction class for M, then for all $d < e$

$$(M, S) \models \forall \varphi \in Q_d(M)[S(\varphi) \Leftrightarrow S(\mathrm{Tr}_d(\varphi))].$$

Proof.

By induction on d. □

Let $S_d = S \cap Q_d(M)$ and let $\Phi(S_d)$ be a (standard) sentence of $(\mathcal{L}_{PA} \cup \{S_d\})(M)$. Then there are $n \in \omega$ and $\Phi^* \in Q_{d+n}(M)$ such that

$$(M, S_d) \models \Phi \Leftrightarrow (M, S) \models S(\Phi^*).$$

Define $(S_d(x))^* = \mathrm{Tr}_d(x)$ and then define Φ^* by induction on complexity of Φ.

Proof:

Lemma

If S is an e-full inductive satisfaction class for M, then for all $d < e$

$$(M, S) \models \forall \varphi \in Q_d(M)[S(\varphi) \Leftrightarrow S(\mathrm{Tr}_d(\varphi))].$$

Proof.

By induction on d. □

Let $S_d = S \cap Q_d(M)$ and let $\Phi(S_d)$ be a (standard) sentence of $(\mathcal{L}_{\mathrm{PA}} \cup \{S_d\})(M)$. Then there are $n \in \omega$ and $\Phi^* \in Q_{d+n}(M)$ such that

$$(M, S_d) \models \Phi \Leftrightarrow (M, S) \models S(\Phi^*).$$

Define $(S_d(x))^* = \mathrm{Tr}_d(x)$ and then define Φ^* by induction on complexity of Φ.

Proof:

Lemma

If S is an e-full inductive satisfaction class for M, then for all $d < e$

$$(M, S) \models \forall \varphi \in Q_d(M)[S(\varphi) \Leftrightarrow S(\mathrm{Tr}_d(\varphi))].$$

Proof.

By induction on d. □

Let $S_d = S \cap Q_d(M)$ and let $\Phi(S_d)$ be a (standard) sentence of $(\mathcal{L}_{\mathrm{PA}} \cup \{S_d\})(M)$. Then there are $n \in \omega$ and $\Phi^* \in Q_{d+n}(M)$ such that

$$(M, S_d) \models \Phi \Leftrightarrow (M, S) \models S(\Phi^*).$$

Define $(S_d(x))^* = \mathrm{Tr}_d(x)$ and then define Φ^* by induction on complexity of Φ.

Fullness of M

Definition

Full(M) $= \{e : M$ has an e-full inductive satisfaction class$\}$.

Proposition

1. Full(M) *is a cut of* M *and, if* Full(M) $> \omega$ *then* M *is recursively saturated.*
2. *If* M *is countable and* Full(M) $>$ Scl(0), *then* Full(M) $= M$.

Theorem (Kaufmann, Schmerl)

There are completions $T \supseteq$ PA, *such that for every* $M \models T$, Full(M) *contains no definable nonstandard elements.*

Fullness of M

Definition

Full(M) $= \{e : M$ has an e-full inductive satisfaction class$\}$.

Proposition

1. Full(M) *is a cut of* M *and, if* Full(M) $> \omega$ *then* M *is recursively saturated.*
2. *If* M *is countable and* Full(M) $>$ Scl(0)*, then* Full(M) $= M$.

Theorem (Kaufmann, Schmerl)

There are completions $T \supseteq$ PA*, such that for every* $M \models T$, Full(M) *contains no definable nonstandard elements.*

Fullness of M

Definition
$\text{Full}(M) = \{e : M \text{ has an } e\text{-full inductive satisfaction class}\}$.

Proposition

1. $\text{Full}(M)$ *is a cut of M and, if* $\text{Full}(M) > \omega$ *then M is recursively saturated.*
2. *If M is countable and* $\text{Full}(M) > \text{Scl}(0)$, *then* $\text{Full}(M) = M$.

Theorem (Kaufmann, Schmerl)
There are completions $T \supseteq \text{PA}$, such that for every $M \models T$, $\text{Full}(M)$ *contains no definable nonstandard elements.*

Fullness of M

Definition
$\text{Full}(M) = \{e : M \text{ has an } e\text{-full inductive satisfaction class}\}$.

Proposition

1. $\text{Full}(M)$ *is a cut of* M *and, if* $\text{Full}(M) > \omega$ *then* M *is recursively saturated.*
2. *If* M *is countable and* $\text{Full}(M) > \text{Scl}(0)$*, then* $\text{Full}(M) = M$.

Theorem (Kaufmann, Schmerl)
There are completions $T \supseteq \text{PA}$*, such that for every* $M \models T$*,* $\text{Full}(M)$ *contains no definable nonstandard elements.*

Problem

Suppose $M \models \mathrm{PA}$ is countable and recursively saturated and $\mathrm{Full}(M) = M$. Does M have a full inductive satisfaction class?

Many satisfaction classes I

Definition
$\mathfrak{A}(X) = \mathrm{card}(\{f(X) : f \in \mathsf{Aut}(M)\})$

Theorem (Krajewski)
Let $S \subseteq M$ be a partial inductive satisfaction class (or a full, not necessarily inductive satisfaction class). Then $\mathfrak{A}(S) = 2^{\aleph_0}$.

Theorem (Schmerl)
If M is countable recursively saturated, and $X \in \mathsf{Class}(M) \setminus \mathsf{Def}(M)$, then $\mathfrak{A}(X) = 2^{\aleph_0}$.

Many satisfaction classes I

Definition

$\mathfrak{A}(X) = \mathrm{card}(\{f(X) : f \in \mathsf{Aut}(M)\})$

Theorem (Krajewski)

Let $S \subseteq M$ be a partial inductive satisfaction class (or a full, not necessarily inductive satisfaction class). Then $\mathfrak{A}(S) = 2^{\aleph_0}$.

Theorem (Schmerl)

If M is countable recursively saturated, and $X \in \mathsf{Class}(M) \setminus \mathsf{Def}(M)$, then $\mathfrak{A}(X) = 2^{\aleph_0}$.

Many satisfaction classes I

Definition

$\mathfrak{A}(X) = \mathrm{card}(\{f(X) : f \in \mathsf{Aut}(M)\})$

Theorem (Krajewski)

Let $S \subseteq M$ be a partial inductive satisfaction class (or a full, not necessarily inductive satisfaction class). Then $\mathfrak{A}(S) = 2^{\aleph_0}$.

Theorem (Schmerl)

If M is countable recursively saturated, and $X \in \mathsf{Class}(M) \setminus \mathsf{Def}(M)$, then $\mathfrak{A}(X) = 2^{\aleph_0}$.

Many satisfaction classes II

Theorem (RK, Kotlarski)

Let M be countable and recursively saturated and let $e \in M$ be nonstandard.

1. *If M has an e-full inductive satisfaction class, then for every $c > \omega$ there are $2^{\aleph_0}$ inductive satisfaction classes, such that any two disagree on a sentence $\varphi < c$.*
2. *If $S \subseteq M$ is an e-full inductive satisfaction class, then there is an e-full inductive satisfaction class $D \subseteq M$ such that for every $c > \omega$ there is an $\varphi < c$ such that $\varphi \in S$ and $\neg\varphi \in D$.*

Many satisfaction classes II

Theorem (RK, Kotlarski)

Let M be countable and recursively saturated and let $e \in M$ be nonstandard.

1. *If M has an e-full inductive satisfaction class, then for every $c > \omega$ there are $2^{\aleph_0}$ inductive satisfaction classes, such that any two disagree on a sentence $\varphi < c$.*
2. *If $S \subseteq M$ is an e-full inductive satisfaction class, then there is an e-full inductive satisfaction class $D \subseteq M$ such that for every $c > \omega$ there is an $\varphi < c$ such that $\varphi \in S$ and $\neg\varphi \in D$.*

Many satisfaction classes III

Theorem (RK)

1. *Let M be countable and recursively saturated. If $e > \omega$ and M has an e-full inductive satisfaction class, then there are $2^{\aleph_0}$ theories* $\mathrm{Th}(M, S)$*, where S is an e-full inductive satisfaction class.*
2. *If $e > \omega$ and S is an e-full inductive satisfaction class, then there are $2^{\aleph_0}$ isomorphism types of expansions (M, D), such that $(M, S, e) \equiv (M, D, e)$.*

Many satisfaction classes III

Theorem (RK)

1. *Let M be countable and recursively saturated. If $e > \omega$ and M has an e-full inductive satisfaction class, then there are $2^{\aleph_0}$ theories* $\mathrm{Th}(M, S)$, *where S is an e-full inductive satisfaction class.*
2. *If $e > \omega$ and S is an e-full inductive satisfaction class, then there are $2^{\aleph_0}$ isomorphism types of expansions (M, D), such that $(M, S, e) \equiv (M, D, e)$.*

Many satisfaction classes IV

Theorem (RK, Schmerl)

Let M be countable and recursively saturated. If $e > \omega$ and M has an e-full inductive satisfaction class, then M has an e-full inductive satisfaction class S such that (M, S) is prime, and in particular (M, S) is rigid.

Theorem (Schmerl)

Let $\mathfrak{A}$ be a linearly ordered structure. Then, for every $M \models \mathrm{PA}$ there is N such that $M \prec_{\mathrm{end}} N$ and $\mathrm{Aut}(\mathfrak{A}) \cong \mathrm{Aut}(N)$.

Corollary

Let $\mathfrak{A}$ be a countable linearly ordered structure and let $M \models \mathrm{PA}$ be countable and recursively saturated. If $e > \omega$ and M has an e-full inductive satisfaction class, then M has an e-full inductive satisfaction class S such that $\mathrm{Aut}(\mathfrak{A}) \cong \mathrm{Aut}(M, S)$.

Many satisfaction classes IV

Theorem (RK, Schmerl)

Let M be countable and recursively saturated. If $e > \omega$ and M has an e-full inductive satisfaction class, then M has an e-full inductive satisfaction class S such that (M, S) is prime, and in particular (M, S) is rigid.

Theorem (Schmerl)

Let $\mathfrak{A}$ be a linearly ordered structure. Then, for every $M \models \mathrm{PA}$ there is N such that $M \prec_{\mathrm{end}} N$ and $\mathrm{Aut}(\mathfrak{A}) \cong \mathrm{Aut}(N)$.

Corollary

Let $\mathfrak{A}$ be a countable linearly ordered structure and let $M \models \mathrm{PA}$ be countable and recursively saturated. If $e > \omega$ and M has an e-full inductive satisfaction class, then M has an e-full inductive satisfaction class S such that $\mathrm{Aut}(\mathfrak{A}) \cong \mathrm{Aut}(M, S)$.

Many satisfaction classes IV

Theorem (RK, Schmerl)

Let M be countable and recursively saturated. If $e > \omega$ and M has an e-full inductive satisfaction class, then M has an e-full inductive satisfaction class S such that (M, S) is prime, and in particular (M, S) is rigid.

Theorem (Schmerl)

Let $\mathfrak{A}$ be a linearly ordered structure. Then, for every $M \models \mathrm{PA}$ there is N such that $M \prec_{\mathrm{end}} N$ and $\mathrm{Aut}(\mathfrak{A}) \cong \mathrm{Aut}(N)$.

Corollary

Let $\mathfrak{A}$ be a countable linearly ordered structure and let $M \models \mathrm{PA}$ be countable and recursively saturated. If $e > \omega$ and M has an e-full inductive satisfaction class, then M has an e-full inductive satisfaction class S such that $\mathrm{Aut}(\mathfrak{A}) \cong \mathrm{Aut}(M, S)$.

Satisfaction classes and automorphisms, a digression

Question

Let $M \models \mathrm{PA}$ be countable and recursively saturated and let $f \in \mathrm{Aut}(M)$. Is there an N such that $M \prec_{\mathrm{end}} N$ and f extends to N? Could there be an f that is not extendible to any elementary end extension?

Proposition

If there is a partial inductive satisfaction class S such that $f \in \mathrm{Aut}(M, S)$, then there is an N such that $M \prec_{\mathrm{end}} N$ and f extends to N.

Proposition

If M is arithmetically saturated then there are $f \in \mathrm{Aut}(M)$ such that $f \notin \mathrm{Aut}(M, S)$ for all partial inductive satisfaction classes S.

Satisfaction classes and automorphisms, a digression

Question

Let $M \models \mathrm{PA}$ be countable and recursively saturated and let $f \in \mathrm{Aut}(M)$. Is there an N such that $M \prec_{\mathrm{end}} N$ and f extends to N? Could there be an f that is not extendible to any elementary end extension?

Proposition

If there is a partial inductive satisfaction class S such that $f \in \mathrm{Aut}(M, S)$, then there is an N such that $M \prec_{\mathrm{end}} N$ and f extends to N.

Proposition

If M is arithmetically saturated then there are $f \in \mathrm{Aut}(M)$ such that $f \notin \mathrm{Aut}(M, S)$ for all partial inductive satisfaction classes S.

Question

Let $M \models \mathrm{PA}$ be countable and recursively saturated and let $f \in \mathrm{Aut}(M)$. Is there an N such that $M \prec_{\mathrm{end}} N$ and f extends to N? Could there be an f that is not extendible to any elementary end extension?

Proposition

If there is a partial inductive satisfaction class S such that $f \in \mathrm{Aut}(M, S)$, then there is an N such that $M \prec_{\mathrm{end}} N$ and f extends to N.

Proposition

If M is arithmetically saturated then there are $f \in \mathrm{Aut}(M)$ such that $f \notin \mathrm{Aut}(M, S)$ for all partial inductive satisfaction classes S.

Satisfaction classes and elementary submodels I

If S is a partial inductive satisfaction class for a model M, then let M_S be the PA-reduct of the smallest elementary submodel of (M, S).

If M_S is not ω, then the restriction of S to M_S is a partial inductive satisfaction class for M_S; hence M_S is recursively saturated.

Question

Let $M \models \mathrm{PA}$ be countable and recursively saturated. For which recursively saturated $M' \prec M$ do there exist partial inductive satisfaction classes S such that $M' = M_S$?.

Satisfaction classes and elementary submodels I

If S is a partial inductive satisfaction class for a model M, then let M_S be the PA-reduct of the smallest elementary submodel of (M, S).
If M_S is not ω, then the restriction of S to M_S is a partial inductive satisfaction class for M_S; hence M_S is recursively saturated.

Question

Let $M \models \mathrm{PA}$ be countable and recursively saturated. For which recursively saturated $M' \prec M$ do there exist partial inductive satisfaction classes S such that $M' = M_S$?.

Satisfaction classes and elementary submodels I

If S is a partial inductive satisfaction class for a model M, then let M_S be the *PA*-reduct of the smallest elementary submodel of (M, S).
If M_S is not ω, then the restriction of S to M_S is a partial inductive satisfaction class for M_S; hence M_S is recursively saturated.

Question

Let $M \models \mathrm{PA}$ be countable and recursively saturated. For which recursively saturated $M' \prec M$ do there exist partial inductive satisfaction classes S such that $M' = M_S$?.

Partial solutions

Theorem (RK)

Let $M \models \mathrm{PA}$ be countable and recursively saturated. If M' is recursively saturated and $M' \prec_{\mathrm{cof}} M$ then there is an inductive satisfaction class S such that $M' = M_S$

Theorem (RK)

If $M \models \mathrm{PA}$ is countable, recursively saturated, but not arithmetically saturated, then there are $M' \prec_{\mathrm{end}} M$ such that M' is recursively saturated and no inductive satisfaction class for M' can be extended to an inductive satisfaction class for M. In particular, for no inductive satisfaction class S for M, $M' = M_S$.

Partial solutions

Theorem (RK)

Let $M \models \mathrm{PA}$ *be countable and recursively saturated. If* M' *is recursively saturated and* $M' \prec_{\mathrm{cof}} M$ *then there is an inductive satisfaction class* S *such that* $M' = M_S$

Theorem (RK)

If $M \models \mathrm{PA}$ *is countable, recursively saturated, but not arithmetically saturated, then there are* $M' \prec_{\mathrm{end}} M$ *such that* M' *is recursively saturated and no inductive satisfaction class for* M' *can be extended to an inductive satisfaction class for* M*. In particular, for no inductive satisfaction class* S *for* M, $M' = M_S$.

Some model theory: Smoryński Stavi Theorem

Theorem

If M is recursively saturated M and $M \prec_{\mathrm{cof}} N$, then N is recursively saturated.

Proof.

It is enough to prove the theorem for countable M. Let S be an partial inductive satisfaction class for M. By the Kotlarski-Schmerl Lemma there is $\bar{S} \subseteq N$ such that $(M, S) \prec (N, \bar{S})$. □

Corollary

Every recursively saturated model of PA *has cofinal recursively saturated elementary extensions of arbitrarily high cardinalities.*

Theorem

If M is recursively saturated M and $M \prec_{\mathrm{cof}} N$, then N is recursively saturated.

Proof.

It is enough to prove the theorem for countable M. Let S be an partial inductive satisfaction class for M. By the Kotlarski-Schmerl Lemma there is $\bar{S} \subseteq N$ such that $(M, S) \prec (N, \bar{S})$. □

Corollary

Every recursively saturated model of PA has cofinal recursively saturated elementary extensions of arbitrarily high cardinalities.

Some model theory: Smoryński Stavi Theorem

Theorem

If M is recursively saturated M and $M \prec_{\text{cof}} N$, then N is recursively saturated.

Proof.

It is enough to prove the theorem for countable M. Let S be an partial inductive satisfaction class for M. By the Kotlarski-Schmerl Lemma there is $\bar{S} \subseteq N$ such that $(M, S) \prec (N, \bar{S})$. □

Corollary

Every recursively saturated model of PA *has cofinal recursively saturated elementary extensions of arbitrarily high cardinalities.*

Theorem

Every countable recursively saturated $M \models \mathrm{PA}$ has a recursively saturated elementary end extension.

Proof.

Let S be an inductive satisfaction class for M and let (N, S') be an elementary end extension of (M, S) given by the MacDowell-Specker Theorem. Then $M \prec_{\mathrm{end}} N$ and N is recursively saturated. □

Remark

1. *Kaufmann model M_K is recursively saturated, but has no recursively saturated elementary end extension.*
2. *Nevertheless, every countable, recursively saturated model M has recursively saturated, κ-like elementary end extensions for all uncountable cardinals κ.*

Theorem

Every countable recursively saturated $M \models \mathrm{PA}$ has a recursively saturated elementary end extension.

Proof.

Let S be an inductive satisfaction class for M and let (N, S') be an elementary end extension of (M, S) given by the MacDowell-Specker Theorem. Then $M \prec_{\mathrm{end}} N$ and N is recursively saturated. □

Remark

1. *Kaufmann model M_K is recursively saturated, but has no recursively saturated elementary end extension.*
2. *Nevertheless, every countable, recursively saturated model M has recursively saturated, κ-like elementary end extensions for all uncountable cardinals κ.*

More model theory: Smoryński's Theorem

Theorem

Every countable recursively saturated $M \models \mathrm{PA}$ has a recursively saturated elementary end extension.

Proof.

Let S be an inductive satisfaction class for M and let (N, S') be an elementary end extension of (M, S) given by the MacDowell-Specker Theorem. Then $M \prec_{\mathrm{end}} N$ and N is recursively saturated. □

Remark

1. *Kaufmann model M_K is recursively saturated, but has no recursively saturated elementary end extension.*
2. *Nevertheless, every countable, recursively saturated model M has recursively saturated, κ-like elementary end extensions for all uncountable cardinals κ.*

A converse to Tarski?

Definition

Let $FS(X)$ be a formula of $\mathcal{L}_{\mathrm{PA}} \cup \{X\}$ expressing that X is a full satisfaction class.

$FS(X)$ is an example of a formula $\Phi(X)$ such that

1. $\mathrm{Con}(\mathrm{PA}^* + \Phi(X))$;
2. If $(M, X) \models \Phi(X)$, then $X \notin \mathrm{Def}(M)$.

Question

Suppose $\Phi(X)$ satisfies 1. and 2. above. Is it true that for every M and $X \subseteq M$, if $(M, X) \models \Phi(X)$, then there is a truth extension $S \in \mathrm{Def}(M, X)$?

A converse to Tarski?

Definition

Let $FS(X)$ be a formula of $\mathcal{L}_{\text{PA}} \cup \{X\}$ expressing that X is a full satisfaction class.

$FS(X)$ is an example of a formula $\Phi(X)$ such that

1. $\mathrm{Con}(\mathsf{PA}^* + \Phi(X))$;
2. If $(M, X) \models \Phi(X)$, then $X \notin \mathrm{Def}(M)$.

Question

Suppose $\Phi(X)$ satisfies 1. and 2. above. Is it true that for every M and $X \subseteq M$, if $(M, X) \models \Phi(X)$, then there is a truth extension $S \in \mathrm{Def}(M, X)$?

A converse to Tarski?

Definition

Let $FS(X)$ be a formula of $\mathcal{L}_{\mathrm{PA}} \cup \{X\}$ expressing that X is a full satisfaction class.

$FS(X)$ is an example of a formula $\Phi(X)$ such that

1. $\mathrm{Con}(\mathsf{PA}^* + \Phi(X))$;
2. If $(M, X) \models \Phi(X)$, then $X \notin \mathrm{Def}(M)$.

Question

Suppose $\Phi(X)$ satisfies 1. and 2. above. Is it true that for every M and $X \subseteq M$, if $(M, X) \models \Phi(X)$, then there is a truth extension $S \in \mathrm{Def}(M, X)$?

A converse to Tarski?

Definition

Let $FS(X)$ be a formula of $\mathcal{L}_{\mathrm{PA}} \cup \{X\}$ expressing that X is a full satisfaction class.

$FS(X)$ is an example of a formula $\Phi(X)$ such that

1. $\mathrm{Con}(\mathsf{PA}^* + \Phi(X))$;
2. If $(M, X) \models \Phi(X)$, then $X \notin \mathrm{Def}(M)$.

Question

Suppose $\Phi(X)$ satisfies 1. and 2. above. Is it true that for every M and $X \subseteq M$, if $(M, X) \models \Phi(X)$, then there is a truth extension $S \in \mathrm{Def}(M, X)$?

Something similar

$FS(X)$ is an example of a formula $\Psi(X)$ such that
If $M \models \mathrm{PA}$ is nonstandard and $(M, X) \models \Psi(X)$, for some $X \subseteq M$, then M is recursively saturated.

Theorem

Suppose $\Psi(X)$ is as above. Then for every M and $X \subseteq M$, if $(M, X) \models \Psi(X) + \mathrm{PA}^*$, then there is a partial inductive satisfaction class S such that $S \in \mathrm{Def}(M, X)$.

Proof.

Let $(M, X) \models \Psi(X) + \mathrm{PA}^*$. Let $(M, X) \prec_{\mathrm{end}} (N, Y)$ and such that $\mathrm{Cod}(N/M) = \mathrm{Def}(M, X)$. In addition, we can assume that N has a partial inductive satisfaction class S'. Then $S = S' \cap M \in \mathrm{Def}(M, X)$ and there is an $e > \omega$ such that $S' \cap Q_e(M)$ is a partial inductive satisfaction class for M. □

Something similar

$FS(X)$ is an example of a formula $\Psi(X)$ such that
If $M \models \mathrm{PA}$ is nonstandard and $(M, X) \models \Psi(X)$, for some $X \subseteq M$, then M is recursively saturated.

Theorem

Suppose $\Psi(X)$ is as above. Then for every M and $X \subseteq M$, if $(M, X) \models \Psi(X) + \mathrm{PA}^*$, then there is a partial inductive satisfaction class S such that $S \in \mathrm{Def}(M, X)$.

Proof.

Let $(M, X) \models \Psi(X) + \mathrm{PA}^*$. Let $(M, X) \prec_{\text{end}} (N, Y)$ and such that $\mathrm{Cod}(N/M) = \mathrm{Def}(M, X)$. In addition, we can assume that N has a partial inductive satisfaction class S'.Then $S = S' \cap M \in \mathrm{Def}(M, X)$ and there is an $e > \omega$ such that $S' \cap Q_e(M)$ is a partial inductive satisfaction class for M. □

Something similar

$FS(X)$ is an example of a formula $\Psi(X)$ such that
If $M \models \mathrm{PA}$ is nonstandard and $(M, X) \models \Psi(X)$, for some $X \subseteq M$, then M is recursively saturated.

Theorem

Suppose $\Psi(X)$ is as above. Then for every M and $X \subseteq M$, if $(M, X) \models \Psi(X) + \mathrm{PA}^*$, then there is a partial inductive satisfaction class S such that $S \in \mathrm{Def}(M, X)$.

Proof.

Let $(M, X) \models \Psi(X) + \mathrm{PA}^*$. Let $(M, X) \prec_{\mathrm{end}} (N, Y)$ and such that $\mathrm{Cod}(N/M) = \mathrm{Def}(M, X)$. In addition, we can assume that N has a partial inductive satisfaction class S'. Then $S = S' \cap M \in \mathrm{Def}(M, X)$ and there is an $e > \omega$ such that $S' \cap Q_e(M)$ is a partial inductive satisfaction class for M. □

Something similar

$FS(X)$ is an example of a formula $\Psi(X)$ such that
If $M \models \mathrm{PA}$ is nonstandard and $(M, X) \models \Psi(X)$, for some $X \subseteq M$, then M is recursively saturated.

Theorem

Suppose $\Psi(X)$ is as above. Then for every M and $X \subseteq M$, if $(M, X) \models \Psi(X) + \mathrm{PA}^*$, then there is a partial inductive satisfaction class S such that $S \in \mathrm{Def}(M, X)$.

Proof.

Let $(M, X) \models \Psi(X) + \mathrm{PA}^*$. Let $(M, X) \prec_{\mathrm{end}} (N, Y)$ and such that $\mathrm{Cod}(N/M) = \mathrm{Def}(M, X)$. In addition, we can assume that N has a partial inductive satisfaction class S'.Then
$S = S' \cap M \in \mathrm{Def}(M, X)$ and there is an $e > \omega$ such that
$S' \cap Q_e(M)$ is a partial inductive satisfaction class for M. □