Revision without ordinals

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Semantic theories of Truth

One goal common to most semantic approaches to Truth is to classify sentences (of a language containing a Truth predicate) into two categories: *paradoxical* and *non-paradoxical* ones. A minimum requirement is that all sentences not involving the Truth predicate will be classified as non-paradoxical, while sentences like the Liar will be classified as paradoxical. Different semantic accounts for Truth can differently classify sentences that lie outside these two extrema. Moreover, one semantic theory often provides a further refinement of both categories as an attempt to explain why one sentence has to fall under one categories rather than another.

Approximation and Revision

Most of the semantic theories of Truth that address this classification issue can be understood as belonging to one of two major competitive methods: *approximation* and *revision*.

Approximation provides a partial interpretation of the Truth predicate starting by a set of non-paradoxical sentences and progressively enlarging it. The sentences that eventually fall into the partial extension of the Truth predicate are those intended to formally capture the notion of non-paradoxicality and are called *grounded* sentences.

Revision provides a formally counterpart of non-paradoxicality via the notion of stability: a sentence (and its negation) is called *stable* if eventually falls into all the extensions of a suitable sequence of total intepretations for the Truth predicate.

Iterations of the jump operator

A central idea, common to both the approximation and the revision approaches, is to build up the extensions of the Truth predicate via a process that iterates an operator Γ on sentences: given a (partial or total) interpretation X for the Truth predicate T, the result $\Gamma(X)$ will still be a set of sentences to be understood as a new interpretation for T. The same for $\Gamma(\Gamma(\mathrm{T}))$ and so on.

Approximation applies Γ to partial interpretations of T and the process will stop when a fixed point of Γ is reached. Revision applies Γ to total interpretations of T and the process never stops, providing a sequence of interpretations suitable to evaluate stability of sentences.

Transfinite iterations

Both approximation and revision need the iteration process to continue into the transfinite. Approximation, because the fixed point of Γ has to be reached at some transfinite stage. Revision, because there is no reason for evaluating stabilities by stopping the process at a finite stage.

A transfinite iteration process can be formalized in a set-theoretical environment, like Zermelo-Fraenkel's, provided that the Trasfinite Recursion theorem is available: we fix a starting point X_0 and a *limit rule* Δ and define a transfinite iteration $s = \langle X_\alpha \mid \alpha \in \mathrm{On} \rangle$ of Γ such that

$$X_{\alpha+1} = \Gamma(X_\alpha) \quad \text{and} \quad X_\delta = \Delta(s \restriction \delta) \text{ for } \delta \text{ limit.}$$

Approximation without ordinals

With minor adjustments, approximation can be recasted as a special case of revision.

As a matter of fact, in the approximation case, transfinite sequences are just *a method of proving* the existence of a fixed point of the operator Γ. The set of the *grounded* sentences can alternatively be defined (and its properties can be proved) without any appeal to transfinite sequences and the Transfinite Recursion theorem. So, even though transfinite sequences are good for formalizing the intuitive idea of an idealized iteration process, the ordinals turn out to play no substantial role in the mathematical notion of groundedness.

Fitting

M. Fitting, *Notes on the Mathematical Aspects of Kripke's Theory of Truth*, 1986:

> *It is most common to establish the existence of smalled fixed points by using ordinally indexed sequences of approximations. This is not necessary, and has two distinct drawbacks: First, it is more mathematical paraphernalia than one needs, tending to obscure the inherent simplicity of the subject. Second, it is a construction that only works for special kinds of fixed points, while in Kripke's theory all fixed points have some role to play.*

Revision without ordinals

The role played by the ordinals seems to make a mathematical distinction between approximation and revision.

So, I think that it is worthy to investigate the possibility that also the revision approach could be formalized in a purely inductive-theoretic way. Namely, that the set of the *stable* sentences could be defined (and its relevant properties could be proved) without any appeal to ordinals. We ask if, even in the revision case, the ordinals just play an instrumental role, so that this apparently mathematical difference between approximation and revision could disappear and we could contrast these two approaches in a mathematical framework which dispenses with ordinals.

Revision sequences

Fix the *revision operator* Γ, a *limit rule* Δ and an *initial guess* h (a characteristic function on Sent), and let $S = S(h, \Gamma, \Delta)$ be the sequence defined as

$$h_0 = h, \quad h_{\alpha+1} = \Gamma(h_\alpha) \quad \text{and} \quad h_\delta = \Delta(S \restriction \delta) \text{ for } \delta \text{ limit.}$$

A sentence σ is *stable in S at* δ if

$$\exists \alpha < \delta \, \forall \beta < \delta \, (\alpha \leq \beta \rightarrow h_\beta(\sigma) = h_\alpha(\sigma))$$

and is *stable in S* if and only if is stable at On.

A sequence $S = S(h, \Gamma, \Delta)$ is a **revision sequence** if h_δ preserves the stable truth value of all sentences which are stable at δ, for all δ limit.

Normal revision sequences

Given a sequence *s* of limit length δ, the *tail* of *s* at $\alpha < \delta$ is: $T^\alpha = \{h_\beta \mid \alpha \leq \beta < \delta\}$.

Let $\liminf(s) = \bigcup\{\bigcap T^\alpha \mid \alpha < \delta\}$.

A sentence σ is stable in S at δ if and only if $\sigma \in \liminf(S \restriction \delta)$.

Def.

A revision sequence $S = S(h, \Gamma, \Delta)$ is a **normal** revision sequence if and only if, for all δ limit,

$$\Delta(S \restriction \delta) = F(\liminf(S \restriction \delta)).$$

A positive result

Thm

For every normal revision sequence S, the set of all sentences that are stable in S is definable by purely inductive-theoretic methods.

Note that both Herzberger sequences and Gupta sequences, as well as approximation sequences, are examples of *normal* revision sequences.

We say that a sentence is *stable in the Herzberger (Gupta) sense* if and only if is stable in every Herzberger (Gupta) sequence. So the notion of *Herzberger stability* (*Gupta stability*) is definable without any appeal to the ordinals.

Iteration without ordinals

The general method of transforming arguments based on Transfinite Recursion in arguments based on Structural Induction applies to a wide range of iterative processes, approximation and (normal) revision included.

This method allows us to eliminate from the proofs three kinds of set-theoretical resources:

- **Ordinals**: we eliminate sequences in favor of *orbits*.
- **Proper classes**: no mention of either formal or informal proper classes is needed.
- **Replacement**: proofs are carried out in Zermelo's set theory (without Choice, Foundation and Replacement)

Orbits vs. sequences

The **orbit** of an operator Γ from a point a is the smallest Γ-closed set containing a as an element. The orbit is the range of the sequence defined by $a_0 = a$ and $a_{n+1} = \Gamma(a_n)$.

The concept of *orbit* captures the *qualitative* part of the information about Γ contained in the sequence.

Given a sequence s of limit length δ, the *head* of s at $\alpha < \delta$ is: $H_\alpha = \{h_\beta \mid \beta < \alpha\}$.

The family $\{H_\delta \mid \delta \text{ limit}\}$ of the limit heads of a *transfinite* sequence S can be seen as a *generalized orbit* which describes the iterative process underlying S in terms of wellorderings and orbits.

Orbits and revision

In the case of a *normal revision sequence* $S = S(h, \Gamma, \Delta)$, the role of the orbits in capturing the relevant information about the revision operator Γ is particularly prominent due to the following facts:

- We can recover $\liminf(S)$ just from the *generalized orbit* of Γ determined by h and Δ.
- We can define the generalized orbit of Γ determined by h and Δ by using only inductive-theoretic methods.

Sets vs.classes - I

Revision sequences are ordinal-lenght sequences, so they are *proper classes*.

The collection of all the Herzberger sequences is a class parametrically defined by a single formula whose parameter is the *initial guess* of the sequence. So we can formally define the notion of *Herzberger stability* by quantifying over this parameter.

It can be shown that *Herzberger stability* can also be defined as stability in all Herzberger's sequences of length ω_1.

The inductive-theoretic reformulation of revision define *Herzberger stability* as stability in all generalized orbits of the revision operator generated by the Herzberger limit rule.

Let H be the formula giving the Herzberger limit rule.

$\text{Stable}_H(\sigma, h)$ means "σ is stable in the Herzberger sequence starting from h".

(Def:) $$\text{Stable}_H(\sigma) \leftrightarrow \forall h\ \text{Stable}_H(\sigma, h).$$

$\text{Stable}(\sigma, s)$ means "σ is stable in the (set) sequence s". $H(s)$ means "s is an ω_1-Herzberger sequence".

(Thm:) $$\text{Stable}_H(\sigma) \leftrightarrow \forall s\ (H(s) \rightarrow \text{Stable}(\sigma, s)).$$

$\text{Stable}(h, \Delta)$ denote "the set of all stable sentences determined by h and Δ". Δ_H is the orbit rule corresponding to H.

(Thm:) $$\text{Stable}_H(\sigma) \leftrightarrow \sigma \in \text{Stable}(h, \Delta_H).$$

C. Kuratowski, *Une méthode d'élimination de nombres transfinis des raisonnements mathémathiques*, 1922:

> *[...] l'existence d'un procédé permettant de supprimer la notion de ces nombres [transfinis] dans les démonstrations des théorèmes qui ne concernent guère le transfini est importante pour les deux raisons suivantes: en raisonnant avec le nombres transfinis on fait implicitement usage de l'axiome de leur existence; or, la réduction du système d'axiomes employés dans les démonstrations est désirable au point de vue logique et mathématiques. En outre, cette réduction affranchit les raisonnements de l'élément qui leur est étranger, ce qui augment leur valeur esthétique.*

K. Kuratowski, *Une méthode d'élimination de nombres transfinis des raisonnements mathémathiques*, 1922:

> *[...] the existence of a process that allows to avoid ordinals, in proving theorems that do not deal with the transfinite, is important for the following two reasons: in reasoning about ordinals we implicitly appeal to axioms that ensure their existence; but to weak the axiom system that we use in proving something is desirable both from a logical and from a mathematical point of view. Moreover, this strategy expunges from the arguments the unnecessary elements, increasing their aesthetic value.*

Sets vs. ordinals

We can see Kuratowski's motivations for eliminating the ordinals from mathematical proofs an instance of the "Purity of method" issue. In his time set theory was a two-sorted theory: sets and ordinals, linked by the Axiom of Ordinals. So, "purity of method" had a precise mathematical sense: if the statement of a theorem is a sentence of the pure language of sets we want to prove it in the same language.

When we identify ordinals with a particular kind of sets, to make a mathematical distinction between a piece of set theory that uses ordinals and one that does not, is much more difficult. Yet, we feel that the two fragments are different.

Nevertheless, the axiomatic motivation given by Kuratowski can still make sense.

Zermelo vs. Fraenkel

The definition of *Herzberger stability* requires some set-theoretical axiom that provides the Axiom of Ordinals as a theorem. This job is usually done by the Replacement axiom.

If we need to use the notion of *Herzberger stability* only to prove some mathematical fact, we can directly define the notion of *stability in all* ω_1*-Herzberger sequences* and derive the result from the latter notion. This eliminate from the argument any reference to the axiom of Replacement.

But the notion of *stability in all* ω_1*-Herzberger sequences* hardly has an autonomous significance separate from the proof of its equivalence with the original notion of Herzberger stability. And proving this equivalence of course requires Replacement. In contrast, *stability in all Herzberger generalized orbits* does have. So Herzberger's revision theory of Truth can be entirely carried out in Zermelo's set theory without Replacement.

Belnap's sequences - A (possible?) limitation

A sentence is *stable in the sense of Belnap* if and only if is stable in all the revision sequences. In this case the definition requires quantification over proper classes, so is not expressible in purely set-theoretical terms. A result by McGee shows that it is possible to formalize the notion of Belnap stability in Zermelo-Fraenkel set theory via an equivalent definition that does not quantify over proper classes.

I guess that it is not possible to do the same in purely inductive-theoretical terms. However, it is possible to formalize the notion of *stable in all normal revision sequences*. Since in a set-theoretical context a revision sequence has to be *definable* by some limit rule, I argue that restricting ourselves to *normal* limit rules (which do not depend on the length of the sequence) is a fair policy when we want to see revision sequences as the formal counterpart of idealized iterative processes.