Change the logic, change the meaning? Quine's dictum, formalism freeness, and inner models from extended logics

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Quine, *Philosophy of Logic* (1970)

"Change the logic, change the meaning."

Quine was referring to a change from classical to intuitionistic logic. (Meaning of the *connectives*.)

Had officially renounced the notion of meaning as unscientific (i.e. unbehavioristic) so remark was a departure.

The concept of meaning may be "fuzzy around the edges" i.e. a criterion of sameness of meaning cannot necessarily be given; but there no reason why principled sufficient conditions cannot be given, namely: change of meaning of a proposition should involve a change in its truth conditions. (Quine never rejected the notion of truth.)

Logical pluralism usually arises in connection with Carnap: “Principle of Tolerance”; Quine (P of L); Beall and Restall look at a particular class of “cases”; (relatively) large class of standard non-classical logics. Adopt a truth-functional notion of meaning. (As opposed to a “meaning as use” account.)

Quine: Change the logic, change the truth conditions of sentences of the logic.

Quine's Dictum, for us: Change the logic, change the model class.

(Example: PA vs PA^2)

Look beyond meaning of the connectives.

Consider rather logical theories and their models.
(Though these do not fall under Beall and Restall’s “cases.”)

What we mean by a change of logic:

Properties that characterize first order logic (Lindström Theorem) **fail**, i.e. failure of compactness and/or Löwenheim-Skolem Theorem. (For constructive logic there are classical, i.e. not constructively valid, proofs of compactness and Löwenheim-Skolem.)

(We may not always have such a measure of change of logic.)

Question

For canonical set-theoretical structures, does a change of logic always involve a change of model class? (Extensionally: change of meaning?)

Our answer: For certain canonical structures, no.

Motivation: Faithfulness

- The question of faithfulness: the realization that there is always a leap of faith involved in formalization; the realization that the gap between our intuitions and their formal counterparts, is unbridgeable.
- That our axiomatizations often turn out to be non-categorical (they have many non-isomorphic models) worsens the problem---in fact it epitomizes the problem.

Incompleteness

"Let us consider the concept of demonstrability. It is well know that whichever way you make it precise by means of a formalism, the contemplation of this very formalism gives rise to new axioms which are exactly as evident as those with which you started, and that this process of extension can be iterated into the transfinite. So there cannot exist any formalism which would embrace all these steps…"

---Gödel, Princeton Bicentennial Remarks, 1946

Entanglement

One seeks a natural, unentangled *mathematical* concept.

Purity issue? "A purity constraint restricts the resources available to solve a problem to those which determine it." (*Purity of Methods*, Arana, Detlefsen, 2011.)

ZFC appears to satisfy a purity constraint in that the entanglement of canonical set-theoretic structures with the underlying logic is apparently minimal. (The epsilon relation is "non-logical." Correspondingly, the entanglement with logic is minimal.)

A few historical examples: 1. Heyting on Brouwer

- "…no formal system can be said to adequately represent an intuitionistic theory. There always remains a residue of ambiguity in the representation of the signs, and it can never be proved with mathematical rigour that the system of axioms really embraces every valid method of proof."
- (Abraham A. Fraenkel and Yehoshua Bar-Hillel. *Foundations of set theory*.)

Heyting on Brouwer cont'

- "…no formalized theory can do justice to intuitive (which is for them intuitionistic) mathematics or any of its subtheories."
- Fraenkel, Bar-Hillel, ibid.

2. Bernays

"It seems in no way appropriate that Cantor's Absolute be identified with set theory formalized in standardized logic, which is considered from a more comprehensive model theory."

---Bernays, Letter to Gödel, 1961. (Collected Works, vol. 4, Oxford)

3. Gödel

Gödel's second monograph on the consistency of the CH, in which the constructible hierarchy is given in a logic free way (by the Gödel operations), **1946 Princeton Bicentennial address**; 1956 Dialectica paper, construction is explicitly "logic free." (Troelstra, introduction).

“Formalism free” means…

- Formalism is taken in the full, modern sense of the term (so *not* thinking of e.g. the axiomatic method of Euclid): Signature, axioms, rules of proof, an associated semantics. (For us: semantics is included!)

Formalism Freeness

Is just the suppression of any of: signature, rules of proof, axioms…

It is a matter of degree (like entanglement).

Indifferentism of the Practice

"Two analysts who wish to collaborate do not need to check whether they were taught the same definition of "real number", as two algebraists do need to check whether they are working with the same definition of "ring.""

---John Burgess, *Putting Structuralism in its Place*, 2004

We can ignore the specific set-theoretical construction in our definitions.

First Order Logic

FOL is the maximal logic satisfying compactness and Downward Lowenheim-Skolem. One can therefore view first order logic purely semantically with no concern as to the syntax. As long as these two model theoretic properties are satisfied, the concept of a definable model class is the same.

Model Theory: *Mathematical*, not *Logical* Properties

- **Tarski**: A class of structures in a finite relational language is universally axiomatizable if and only if it is closed under isomorphism, substructure and if for every finite substructure B of a structure A, B $\in$ K then A $\in$ K.
- **Birkhoff**: A class K of algebras is axiomatized by a set of equations if and only it it is closed under homomorphism, subalgebra, and direct product.

Contemporary Model Theory: AEC's

- Shelah, develops the model theory of (large) infinitary languages, then dispenses with language, merely stating the (mathematical) properties needed.
- The connection to infinitary languages was then forgotten and AECs are studied on their own. One wishes to prove categoricity theorems. It was essential to have the language in the background, but then language could be dispensed with.
- Trend in CMT towards the suppression of formulas.

AEC's

Abstract Elementary Classes: closure under unions of chains, with respect to an abstract "submodel relation" (mimics elementary submodel relation); other purely mathematical properties: JEP and Amalgamation.

Baldwin’s analysis of theorems about AEC’s

- Shelah’s Presentation theorem: “passing through the syntax, Shelah obtains a purely semantic theorem.”
- The syntactic condition in the theorem is a set of sentences in roughly Tarski’s sense….but we are able to deduce purely semantical conclusions.

(If an AEC with Lowenheim number $\aleph_0$ has a model of cardinality ω_1 it has arbitrarily large models.)

Quasiminimal excellent classes of Zilber

"Zilber's notion of a *quasiminimal excellent class* was developed to provide a smooth framework for proving the categoricity in all uncountable powers of Zilber's pseudo-exponential field. This example itself is developed in a standard model theoretic framework in $L\omega 1$,ω (Q)….The fundamental result that a quasiminimal excellent class is categorical in all uncountable powers can be presented in a formalism-free way. The key point is that there are no axioms in the object language of the general quasiminimal excellence theorem; there are only statements about the combinatorial geometry determined by what are in the application the ($L\omega 1$,ω)-definable sets."---Baldwin, ibid

"…while formalization is the key tool for the general foundational analysis and has had significant impact as a mathematical tool, there are specific problems in mathematical logic (Section 1.3) and philosophy (Section 2) where 'formalism-free' methods are essential."

---J. Baldwin, *Formalization, Primitive Concepts, and Purity, 2011*

Recursion Theory

"Although Carol Karp appreciated recursive function theory, she disliked proofs which involved codings and systems of notations. In her work on infinitary set theory she noticed that infinitely long formulae sometimes allowed her to circumvent notations…She discovered that by varying the logic in the system one could get a host of results about recursion theory and its extensions; furthermore it could be done without any ad hoc notations. Unfortunately, she only had time to work out some of the details for fragments of infinitary languages of the form Law (i.e., finite-quantifier infinitary languages)."

It was her research on the infinitely long formal proofs that led Karp to the concept of L-R.E. on A. However, it is clear that the actual structure of the proofs is irrelevant, for all that is ever used is the consequence relation. Thus, for the purpose of discussing extensions of recursion theory, it does not make much sense to dwell too much upon the axioms and rules of inference. Consistency properties are a natural way of getting all the benefits of completeness while, at the same time, avoiding formal proofs. (Lopez-Escobar)

Väänänen: Mathematics altogether is indifferent to a choice of logic, especially when that choice is between first order set theory and second order logic. From the practical point of view, the working mathematician will—and should—be indifferent to the choice between FO and SO, and there are deep theoretical reasons why this should be the case. (BSL, 2000, 2012)

Both logics exhibit the same degree of (internal) categoricity, or failure of the same, on close inspection.

“Unreasonable” effectiveness of semantic methods:

- Theorem (Väänänen-Vardi, Gödel, Parikh): Given a concept of provability in predicate logic, there is no recursive function f such that for all φ that are valid, if the length of the proof (in set theory) of validity of φ is n, then the length of the predicate logic proof of φ is at most f(n).

Naturalize Content

Arana: classification of formal and informal content.

Can a notion of set-theoretical content be isolated?

We look at a specific case.

Part 2: Gödel’s Princeton Bicentennial Lecture, 1946

“Tarski has sketched in his lecture the great importance (and I think justly) of the concept of general recursiveness (or Turing computability). It seems to me that this importance is largely due to the fact that with this concept one has succeeded in giving a absolute definition of an interesting epistemological notion, i.e. one not depending on the formalism chosen.”

"In all other cases treated previously, such as demonstrability or definability, one has been able to define them only relative to a given language, and for each individual language it is clear that the one thus obtained is not the one looked for."

E.g. being definable in set theory is not definable. "Take the least undefinable ordinal…"

This, I think, should encourage one to expect the same thing to be possible also in other cases (such as demonstrability or definability). It is true that for these other cases there exist certain negative results, such as the incompleteness of every formalism. . . But close examination shows that these results do not make a definition of the absolute notions concerned impossible under all circumstances, but only exclude certain ways of defining them, or at least, that certain very closely related concepts may be definable in an absolute sense.[42]

Gödel Princeton Bicentennial lecture 1946

Gödel: Intuitive concept (of definability) to be made precise: “Comprehensibility by our mind.”

Part 2: Implementation

3 epistemological notions: computability, provability, definability.

Each come with their own paradoxes.

For each notion, we want transcendence (of a kind)---but we also wish to avoid undefinability in set theory.

Gödel’s two notions of definability

- Two canonical inner models:
  - Constructible sets
    - Model of ZFC
    - Model of GCH
    - *Definable*
  - Hereditarily ordinal definable sets
    - Model of ZFC
    - CH? – independent
    - *Definable* (Levy Reflection)

Constructibility

- Constructible sets (L):

$$
\begin{aligned}
L_0 &= \emptyset \\
L_{\alpha+1} &= \mathrm{Def}(L_\alpha) \\
L_\nu &= \textstyle\bigcup_{\alpha<\nu} L_\alpha \text{ for limit } \nu
\end{aligned}
$$

Ordinal definability

- Hereditarily ordinal definable sets (HOD):
- Take the ordinals as primitive terms.
  - A set is ordinal definable if it is of the form

$$\{a : \varphi(a,\alpha_1,\ldots, \alpha_n)\}$$

where $\varphi(x,y_1,\ldots, y_n)$ is a first order formula of set theory.

  - A set is hereditarily ordinal definable if it and all elements of its transitive closure are ordinal definable.

- Myhill-Scott: Hereditarily ordinal definable sets (HOD) can be seen as the constructible hierarchy based on second order logic (in place of first order logic):

$$\begin{array}{lcl} L'_0 & = & \emptyset \\ L'_{\alpha+1} & = & \mathrm{Def}_{SOL}(L'_\alpha) \\ L'_\nu & = & \bigcup_{\alpha<\nu} L'_\alpha \text{ for limit } \nu \end{array}$$

- Myhill-Scott: Use the *real* power set operation in your definitions. (I.e. the one in V.)
- Chang considered a similar construction with the infinitary logic $\mathcal{L}_{\omega_1\omega_1}$ in place of first order logic.

But this does not satisfy choice (if we assume uncountably many measurable cardinals). It is *not* a fragment of SOL (for cardinality reasons: SOL has only countable many formulas, whereas in the Chang model you can define any countable structure).

C($\mathcal{L}$*)

- $\mathcal{L}$* any logic. We define C($\mathcal{L}$*):

$$\begin{array}{lcl} L'_0 & = & \emptyset \\ L'_{\alpha+1} & = & \mathrm{Def}_{\mathcal{L}^*}(L'_\alpha) \\ L'_\nu & = & \bigcup_{\alpha<\nu} L'_\alpha \text{ for limit } \nu \end{array}$$

- C($\mathcal{L}$*) = the union of all L'_α

- If V=L, then V=HOD=Chang's model=L.
- If there are uncountably many measurable cardinals then AC fails in the Chang model. (Kunen.)

Joint work with M. Magidor and J. Väänänen

Looking ahead:

- For a variety of logics C($\mathcal{L}^*$)=L
  - Gödel's L is very robust, not limited to first order logic
- For a variety of logics C($\mathcal{L}^*$)=HOD
  - Gödel's HOD is robust, not limited to second order logic
- For some logics C($\mathcal{L}^*$) is a potentially interesting new inner model.

Robustness of L

- $Q_1 x\varphi(x) \Leftrightarrow \{a : \varphi(a)\}$ is uncountable
- $C(\mathcal{L}(Q_1)) = L$.
- In fact: $C(\mathcal{L}(Q_\alpha)) = L$, where
  - $Q_\alpha x\varphi(x) \Leftrightarrow |\{a : \varphi(a)\}| \geq \aleph_\alpha$
- Other logics, e.g.

  weak second order logic, ``absolute" logics, etc.

Robustness of L (contd.)

- A logic $\mathcal{L}^*$ is absolute if ``$\varphi \in \mathcal{L}^*$'' is $\Sigma_1$ in $\varphi$ and ``M$\models\varphi$'' is Δ_1 in M and φ in ZFC.
  - First order logic
  - Weak second order logic
  - $\mathcal{L}(Q_0)$: ``there exists infinitely many
  - Finite fragments of $\mathcal{L}_{\omega_1\omega}$, $\mathcal{L}_{\infty\omega}$: infinitary logic
  - Finite fragments of $\mathcal{L}_{\omega_1 G}$, $\mathcal{L}_{\infty G}$: game quantifier logic

HOD: What Myhill-Scott really prove

- In second order logic $\mathcal{L}^2$ one can quantify over arbitrary subsets of the domain.
- A more general logic $\mathcal{L}^{2,F}$: in domain M can quantifier only over subsets of cardinality κ with $F(\kappa) \leq |M|$.
- F any function, e.g. $F(\kappa)=\kappa$, κ^+, 2^κ, $\beth_\kappa$, etc

Theorem

- For all F: $C(\mathcal{L}^{2,F})$=HOD
- Third, fourth order, etc logics give HOD. (Definability reasons)

Observations: avoiding L

- $C(\mathcal{L}_{\omega_1\omega}) = L(\mathsf{R})$

(every formula in lhs can be coded by a real; every real can be coded by a formula of lhs.)

- $C(\mathcal{L}_{\infty\omega}) = V$ (same as above, but for sets)

Generalized Quantifiers

- $Q_1^{MM}xy\varphi(x,y) \Leftrightarrow$ there is an uncountable X such that $\varphi(a,b)$ for all a,b in X
  - Can express Suslinity of a tree. (No uncountable branches, no uncountable antichains)
  - Can be badly incompact; is countably compact (i.e. w.r.t. countable theories) if V=L. L-Skolem down to $\aleph_1$.
- $Q_0^{cf}xy\varphi(x,y) \Leftrightarrow \{(a,b) : \varphi(a,b)\}$ is a linear order of cofinality ω
  - Fully compact extension of first order logic. (Whatever the size of the vocabulary, if a theory of this logic is finitely consistent, then it is consistent.) L-Skolem down to $\aleph_1$.

- aa logic, Hartig quantifier
- Cofinality was essential in Shelah’s *provable* results on size of the continuum.
- In CMT, both generalized quantifiers and infinitary languages have reemerged, due to work of Zilber and others.

Theorems

$C(\mathcal{L}(Q_1^{MM})) = L$, assuming $0^\#$.

Why? If there is an uncountable homogeneous set in V (w.r.t. a definable relation) then there is one in L. Roughly follows from the fact that ω_1 is weakly compact in L.

So assuming large cardinals, L "reads" $\mathcal{L}(Q_1^{MM})$ as first order.

Consistent: $C(\mathcal{L}(Q_1^{MM})) \neq L$ (forcing construction due to Jensen)

Theorems

- $C(\mathcal{L}(Q_0^{cf})) \neq L$, assuming $0^{\#}$.
- Proof depends on: if α is regular in L and cofinality of α is >ω, we can express this in $C(\mathcal{L}(Q_0^{cf}))$. But then α belongs to the set of canonical indiscernibles, i.e we can define $0^{\#}$ in $C(\mathcal{L}(Q_0^{cf}))$.

Theorems

- $C(\mathcal{L}(Q_0^{cf}))$ contains the Dodd-Jensen Core Model, same for the Härtig quantifier. (Härtig-L "sees" the ultrafilter which generates the iterated mouse. From this we get the original mouse. So all mice all present.)

- $C(\mathcal{L}(Q_0^{cf}))$ contains L^μ, if L^μ exists. Uses Dodd-Jensen Covering Lemma.

More theorems

- If $V=C(\mathcal{L}(Q_0^{cf}))$ then continuum is at most $\omega_{2,}$ and there are no measurable cardinals.

Proof : $V=C(L(Q_0^{cf}))$ implies continuum is at most ω_2.

- Condensation argument.
- If *r* is a real, then *r* is in some $X \prec C(\mathcal{L}(Q_0^{cf}))$ such that X ``knows" about cofinality ω. Need witnesses both for cofinality ω and for cofinality greater than ω. In latter case we change the higher cofinalities to cofinality ω_1 by a chain argument.
- X then has cardinality ω_1.
- Then X isom. to L'_α for some α, $\alpha<\omega_2$
- Thus there are at most ω_2 reals.
- Consistently: exactly ω_2 reals.

A real is always constructed on levels of rank less than ω_2.

$V=C(L(Q_0^{cf}))$ implies that there are no measurable cardinals.

- Suppose $i:V \rightarrow M$, κ first ordinal moved, M closed under κ-sequences.
- $(C(\mathcal{L}(Q_0^{cf}))^M=C(\mathcal{L}(Q_0^{cf}))$, since M and V have the same ω-cofinal ordinals (since they have the same ω-sequences).
- So M=V.
- $i:V \rightarrow V$, κ first ordinal moved
- Contradiction! (By Kunen.)
- This is like same proof for V=L.
- Smaller large cardinals are consistent with V=L, hence with $V=C(\mathcal{L}(Q_0^{cf}))$.

More theorems

If there is a Woodin cardinal, then ω_1 is inaccessible in $C(\mathcal{L}(Q_0^{cf}))$. (Stationary tower forcing. Gives an embedding into a model which is closed under ω sequences, moving ω_1 to the Woodin cardinal. Then $(C(\mathcal{L}(Q_0^{cf}))^M = (C(\mathcal{L}(Q_{<\lambda}^{cf}))^V$.

Actually Mahlo.

Theorem 4.7. *If there is a Woodin cardinal, then $\aleph_1$ is inaccessible in $C(Q^{\mathrm{cf}}_\omega)$.*

Proof. Suppose $\aleph_1$ is the successor of α in $C(Q^{\mathrm{cf}}_\omega)$. Let λ be Woodin, $\mathbb{Q}_{<\lambda}$ the countable stationary tower forcing and G generic for this forcing. In $V[G]$ there is $j : V \to M$ with $M^\omega \subset M$ and $j(\omega_1) = \lambda$. Since

$$V \models \omega_1^V = (\alpha^+)^{C(Q^{\mathrm{cf}}_\omega)},$$

we have

$$M \models \lambda = (\alpha^+)^{C(Q^{\mathrm{cf}}_\omega)}.$$

But $C(Q^{\mathrm{cf}}_\omega)^M = C(Q^{\mathrm{cf}}_\omega)^{V[G]}$ as $M^\omega \subseteq M$. Note also that $C(Q^{\mathrm{cf}}_\omega)^{V[G]} = C(Q^{\mathrm{cf}}_{<\lambda})^V$. So

$$M \models \lambda = (\alpha^+)^{(C(Q^{\mathrm{cf}}_\omega)^{V[G]})} = (\alpha^+)^{(C(Q^{\mathrm{cf}}_{<\lambda})^V)} \leq (\alpha^+)^V \leq \omega_1^V,$$

a contradiction. □

Generic absoluteness

Suppose there is a proper class of Woodin cardinals. Then:

- Truth in $C(\mathcal{L}(Q_\alpha^{cf}))$ is forcing absolute and independent of α. (Stationary tower forcing again.)
- Cardinals $>\omega_1$ are all indiscernible for $C(\mathcal{L}(Q_0^{cf}))$. (Another STF.)
- Is CH true in $C(\mathcal{L}(Q_\alpha^{cf}))$? This is forcing absolute and independent of α.

Suppose there is a proper class of Woodin cardinals. Suppose $\mathcal{P}$ is a forcing notion and $G \subseteq \mathcal{P}$ is generic. Denote $C^* = C(Q_\omega^{\mathrm{cf}})$.

Claim: $Th((C^*)^V) = Th((C^*)^{V[G]})$

Proof. Let G be $\mathcal{P}$-generic. Let us choose a Woodin cardinal $\lambda > |\mathcal{P}|$. Let H_1 be generic for the countable stationary tower forcing $\mathbb{Q}_{<\lambda}$. In $V[H_1]$ there is a generic embedding $j_1 : V \to M_1$ such that $V[H_1] \models M_1^\omega \subseteq M_1$ and $j(\omega_1) = \lambda$. Hence $(C^*)^{V[H_1]} = (C^*)^{M_1}$ and

$$j_1 : (C^*)^V \to (C^*)^{M_1} = (C^*)^{V[H_1]} = (C^*_{<\lambda})^V.$$

and therefore by elementarity $Th((C^*)^V) = Th((C^*_{<\lambda})^V)$.

Since $|\mathbb{Q}| < \lambda$, λ is still Woodin in $V[G]$. Let H_2 be generic for the countable stationary tower forcing $\mathbb{Q}_{<\lambda}$ over $V[G]$. Let $j_2 : V[G] \to M_2$ be the generic embedding. Now $V[H_2] \models M_2^\omega \subseteq M_2$ and $j_2(\omega_1) = \lambda$. Hence

$$j_2 : (C^*)^{V[G]} \to (C^*)^{M_2} = (C^*)^{V[H_2]} = (C^*_{<\lambda})^{V[G]} = (C^*_{<\lambda})^V.$$

and therefore by elementarity $(C^*)^V \equiv (C^*_{<\lambda})^V \equiv (C^*)^{V[G]}$.

$\square$

Avoiding HOD

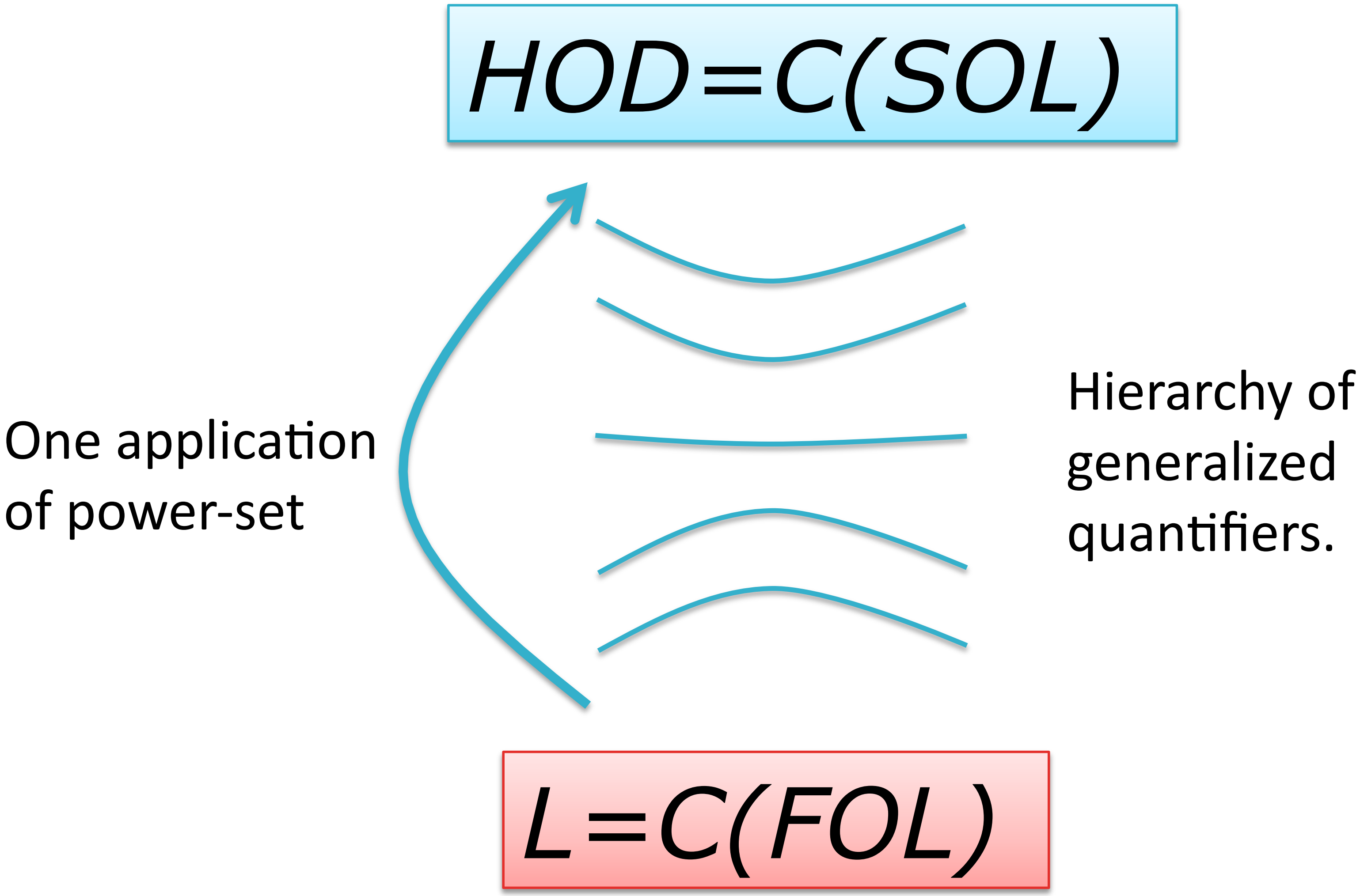
- $C(L(Q_0^{cf})) \neq$ HOD if there are uncountably many measurable cardinals.

There is some countable sequence which is not in $C(L(Q_0^{cf}))$, but can be chosen to be in HOD.

(Like Kunen's proof that uncountably many measurables imply failure of Axiom of Choice in the Chang Model (= $C(L_{\omega_1\omega_1})$).)

- The passage from L to HOD involves one application of the power set operation to the underlying logic.

- We wish to understand the intermediate idealizations involved.


HOD=C(SOL)
One application
of power-set
Hierarchy of
generalized
quantifiers.
L=C(FOL)

Semantic extensions of ZFC

- Replace FOL by an extension of it in the separation and replacement axioms of ZFC.
- Around $L(Q_0)$: exactly omega-standard models of ZFC. (Can define the standard natural numbers in the model.) Weak second order etc…
- Around $L(Q_0^{MM})$: exactly transitive models of ZFC. (Can define well-foundedness.)
- Between $L\omega_1\omega$ and $L\omega_1\omega_1$: exactly countably closed transitive models of ZFC.

Task: Reals

Vary the logic in Kleene's Ramified Analytic Hierarchy (Kleene 1959).

Naturalizing content again

- We have defined an equivalence relation on extensions of first order logic relative to an inner model construction.
- This seems to be a general method, i.e. applicable to various classes of logics relative to various structures.

What “naturalize” means

- Look for heuristic principles rather than embed the problem into some suitable formal system.

E.g.: “Content and extension stand in inverse proportion to eachother.” ---Kant; Port Royal

(Bolzano: slogan does not stand up under criticism)

- We usually think of categoricity as indicating the presence of content (of the theory).
- Here we consider uniqueness relative to classes of logics, a different indication of content.

Philosophy

“The world shows up for us.”

---Alva Noe

Thank you!