Adding standardness to nonstandard models

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Introduction

- Let $M \models \mathrm{PA}$ be nonstandard.
- The set of standard natural numbers ω is an initial segment.
- We may add a unary predicate ω for this, and obtain (M, ω).
- Do structures of the form (M, ω) have interesting model theory?

Background

- *Henkin–Orey theorem*. Model theory of the omega rule:

$$\frac{\phi(0), \phi(1), \ldots, \phi(k), \ldots}{\forall x \in \omega \ \phi(x)}$$

- Kanovei: *On external Scott algebras in nonstandard models of Peano arithmetic*, JSL 1996. Characterises for $M \prec \omega$ the algebras

$$\mathrm{Rep}(M, \omega) = \{A \subseteq \omega : A \text{ is 0-definable in } (M, \omega)\}$$

- Kaye, Kossak, and Wong: *Adding standardness to nonstandard arithmetic*. To appear.

Background, continued

- General background in models of PA. Many structural properties of M (e.g. strength of ω) can be expressed in a first order way in (M, ω).
- Omitting types. Engström–Kaye theory of *transplendence*. NDJFL, Just appeared.
- $\mathrm{SSy}(M) = \{A \subseteq \omega : A$ is def'd in (M, ω) by $x \in \omega \wedge \theta(x, a)\}$
- Encoding second order systems. (M, ω) interprets $(\omega, \mathrm{SSy}(M))$, a model of at least WKL_0.

Interpretation of second order arithmetic

- Replace number quantifiers $\forall x \ldots$ with $\forall x \in \omega \ldots$.
- Replace set quantifiers $\forall A \ldots u \in A \ldots$ with

$$\forall a \ldots (a)_u \neq 0 \ldots.$$

- Everything else stays the same.
- (M, ω) interprets $(\omega, \mathrm{SSy}(M))$, which can be arbitrarily strong.

Definitions

- $\mathrm{SSy}(M)$ is the set of $A \subseteq \omega$ coded in M. $\mathrm{Rep}(M)$ is the parameter-free version.
- $\mathrm{SSy}(M, \omega)$ and $\mathrm{Rep}(M, \omega)$ are similar, but for the expanded language.
- M is *full* if $\mathrm{SSy}(M) = \mathrm{SSy}(M, \omega)$.
- M is *semi-full* if there is $\theta(x, v)$ such that

$$\mathrm{SSy}(M, \omega) = \{A \subseteq \omega : A = \{x : \theta(x, a)\} \text{ some } a \in M\}$$

- M is *fully saturated* if M is full and recursively saturated.

Truth 1

Let $M \models \mathrm{PA}$ and $K \subseteq M$, possibly but not necessarily definable in (M, ω). A *certificate* of truth in K is an M-finite set of triples $\langle \phi, a, t \rangle$ such that

- $\langle \phi, a, t \rangle \in c \Rightarrow \langle \phi, a, 1-t \rangle \notin c$
- $\langle \neg\phi, a, t \rangle \in c \Rightarrow \langle \phi, a, 1-t \rangle \in c$
- $\langle \phi \wedge \psi, a, 1 \rangle \in c \Rightarrow \langle \phi, a, 1 \rangle \in c$ and $\langle \psi, a, 1 \rangle \in c$
- $\langle \phi \wedge \psi, a, 0 \rangle \in c \Rightarrow \langle \phi, a, 0 \rangle \in c$ or $\langle \psi, a, 0 \rangle \in c$
- $\langle \forall x_i\ \phi, a, 1 \rangle \in c \Rightarrow \forall b \in K\ \langle \phi, a[b/i], 1 \rangle \in c$
- $\langle \forall x_i\ \phi, a, 0 \rangle \in c \Rightarrow \exists b \in K\ \langle \phi, a[b/i], 0 \rangle \in c$
- $\langle \phi, a, 1 \rangle \in c \Rightarrow K \models \phi[a]$, when ϕ is atomic
- $\langle \phi, a, 0 \rangle \in c \Rightarrow K \not\models \phi[a]$, when ϕ is atomic

Truth 2

- c is *K-complete* if it is uniform in K: if some formula is decided by c then all K-substitution instances of it are decided too.
- Two K-complete certificates agree on any formula they decide.
- 'there is a K-complete certificate making ϕ true' would be a definition of truth for K in (M, K, ω) provided enough certificates exist.

Strength

- Suppose $K = \omega$ is strong in M (i.e. $\omega, \mathrm{SSy}(M) \vDash \mathrm{ACA}_0$). Then there are enough K-certificates.
- Proof is by induction in the meta-theory.
- Application (KKW). Let $M \vDash \mathrm{PA}$ be a nonstandard model, and ω is strong in M . Let (K, N) be a model of $\mathrm{Th}(M, \omega)$ where N is nonstandard. Then N has a full inductive satisfaction class. In particular, N is recursively saturated.

Elementary substructures

- Suppose $K \prec M$ is not cofinal. Then there are enough K-certificates.
- Proof is by induction in the meta-theory.
- Application (Kanovei). If $\omega \prec M$ then $\mathrm{Th}(\omega, +, \cdot) \in \mathrm{Rep}(M, \omega)$.

The theory

- Since $\mathrm{Th}(M, \omega)$ encodes strong second order arithmetic as well as many model theoretic properties, results can only be relative to what is known about second order arithmetic.
- For a given T extending PA there are continuum-many theories $\mathrm{Th}(M, \omega)$ with $M \vDash T$.
- But, given T, there is a canonical $\mathrm{Th}(M, \omega)$ for some $M \vDash T$.

The canonical completion

- If M_1, M_2 are ω-saturated then $M_1 \equiv_{\omega_1,\omega} M_2$ hence $(M_1, \omega) \equiv (M_2, \omega)$.
- Hence $T^\omega = \mathrm{Th}(M, \omega)$ where $M \models T$ is ω-saturated does not depend on M.
- More generally, for $\bar{a} \in N \models \mathrm{PA}$, let $\mathrm{tp}^\omega(\bar{a})$ be the canonical completion of $\mathrm{tp}(\bar{a})$ to $\mathscr{L}_\mathrm{A}, \omega, \bar{a}$.

ω-elementary models

- N is *ω-elementary* if $(N, \bar{a}, \omega) \models \mathrm{tp}^{\omega}(\bar{a})$ for all $\bar{a}$.
- Equivalently, N is ω-elementary if $(N, \omega) \prec (M, \omega)$ for some ω-saturated M.
- Countable ω-elementary models exist by the Löwenheim–Skolem Theorem.

Transplendent models

- Kaye–Engström: A model M is *transplendent* if it has expansions to any coded $T + p\!\uparrow$ that is consistent with $\mathrm{Th}(M)$ in an ω-saturated model.
- Transplendent models of PA are ω-elementary.
- If N is ω-elementary then N is full.
- If N is ω-elementary then $(\omega, \mathrm{SSy}(N)) \prec (\omega, \mathscr{P}(\omega))$.
- *Are ω-elementary models of PA transplendent?*

Semi-full models

This section makes some progress on $\mathrm{SSy}(M, \omega)$ by looking at interpretations between first order arithmetic with ω and second order arithmetic.

- (M, ω) interprets $(\omega, \mathrm{SSy}(M))$
- In fact, if M is semi-full, (M, ω) interprets $(\omega, \mathrm{SSy}(M, \omega))$

$$\forall A \ldots u \in A \ldots \text{ is replaced by } \forall a \ldots \theta(u, a, \omega) \ldots$$

- Corollary: if M is semi-full then $(\omega, \mathrm{SSy}(M, \omega)) \models \mathrm{CA}_0$.

For $k \in \omega$ we define a family of *partial* translations τk from $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ to $\mathscr{L}_{\mathrm{II}}$ describing properties on (M, ω) in terms of $(\omega, \mathrm{SSy}(M))$.

- $A_k^{\bar{a}}$ is the set $\Sigma_k{-}\mathrm{tp}(\bar{a})$ of all Gödel numbers of Σ_k formulas true of $\bar{a}$ in M.
- $(\psi(\bar{n}, \bar{a}))^{\tau k}$ is the formula

$$\psi(\mathrm{clterm}(\bar{n}), x_1, \ldots, x_n) \in A_k^{\bar{a}},$$

  defined when k is sufficiently large.
- $(\forall b\ \psi(\bar{n}, \bar{a}, b))^{\tau k}$ is

$$\forall A_k^{\bar{a},b}\ (A_k^{\bar{a},b} \text{ extends } A_k^{\bar{a}} \rightarrow (\psi(\bar{n}, \bar{a}, b))^{\tau k}),$$

  where this is defined.

Interpreting (M, ω), continued

- The interpretation commutes with usual boolean connectives, etc.
- Nonstandard models of PA are weakly saturated, and all Σ_k, Π_k types are coded.
- Nevertheless, it seems that the interpretation is ‘local’: sufficiently large k should be chosen for the formula in question.

Full models

- Theorem: $M \models \mathrm{PA}$ is full if and only if $(\omega, \mathrm{SSy}(M)) \models \mathrm{CA}_0$.
- Proof: one direction has been done; for the other, translate $\theta(x, a, \omega)$ defining $A \in \mathrm{SSy}(M, \omega)$ into second order logic and apply comprehension.
- There are full models $N \models \mathrm{PA}$ (indeed, recursively saturated ones) for which $(\omega, \mathrm{SSy}(N))$ is not a β-model.

A sufficient condition for $\mathrm{SSy}(M, \omega)$

- Theorem: If a countable Scott set $\mathscr{X}$ has $(\omega, \mathscr{X}) \models \mathrm{CA}_0$ then it is $\mathrm{SSy}(M, \omega)$ for some M.
- In fact we may take M to be fully saturated here,...
- ...or alternatively we may take M to be prime so that $\mathscr{X} = \mathrm{Rep}(M) = \mathrm{Rep}(M, \omega)$.

- If $M \models \mathrm{PA}$ is nonstandard and $\mathscr{X} = \mathrm{SSy}(M)$ then $\mathrm{SSy}(M, \omega) = \mathrm{Def}(\omega, \mathscr{X})$, the set of sets $A \subseteq \omega$ definable in $\mathrm{Def}(\omega, \mathscr{X})$ (with parameters).
- Note that the comprehension scheme (CA_0) says that $\mathscr{X} = \mathrm{Def}(\omega, \mathscr{X})$, but this is not true for all $\mathscr{X}$.

Prime models

- Scott: if $(\omega, \mathscr{X}) \models \mathrm{WKL}_0$ there is $M \models \mathrm{PA}$ such that $\mathrm{Rep}(M) = \mathrm{SSy}(M) = \mathscr{X}$.
- There are such $\mathscr{X}$ such that each $A \in \mathscr{X}$ is Π^0_∞
- Let $M \models \mathrm{PA}$ be prime such that each $A \in \mathrm{SSy}(M)$ is Π^0_∞. Then $\mathrm{SSy}(M, \omega) = \Pi^0_\infty$.
- Hence there are models $M \models \mathrm{PA}$ with $(\omega, \mathrm{SSy}(M, \omega)) \not\models \mathrm{CA}_0$ and M is not semi-full.
- In general truth on ω is not definable in (M, ω) when M is not a model of true arithmetic.