Truth, Conditionals, and Paradox

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Numbers and Truth
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Overview

Field's wish list

1. a *real conditional* ($\mapsto$) that validates:
   - the unrestricted Tarski-biconditionals (with the Tarski-biconditionals spelled out in terms of the "real conditional")
   - most of the principles that we want for conditionals
2. the *semantical deficiency* of paradoxical sentences must be validly expressible in the object language.

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   - most of the principles that we want for conditionals
2. the *semantical deficiency* of paradoxical sentences must be validly expressible in the object language.

Fact: The material implication of Kripke's theory of truth does not satisfy these requirements.

Field's conditional 1

Field defines a semantics for a primitive conditional operator $\mapsto$ by interleaving the Kripke jump and a revision operator

- Start with an arbitrary extension of $\mapsto$ and build a Kripkean least fixed point for T keeping the interpretation of $\mapsto$ fixed
- Set $\langle \phi, \psi \rangle$ in the extension of $\mapsto$ if $V(\phi) \leq V(\psi)$; put $\langle \phi, \psi \rangle$ in the anti-extension of $\mapsto$ otherwise
- Build a Kripkean least fixed point for the truth predicate
- ...
- For limit ordinals, take the *liminf* rule for defining extension and anti-extension of the truth predicate. Set $\langle \phi, \psi \rangle$ in the extension of $\mapsto$ if $V(\phi) \leq V(\psi)$ cofinally before; similarly for anti-extention
- ...

Field's conditional 2

Proposition

Field's hierarchy of models does not reach a fixed point.

Definition

The *ultimate truth value* of ϕ is the co-final value of ϕ in this hierarchy of stages.

Virtues

- the Field-conditional satisfies the unrestricted Tarski-biconditionals
- the Field-conditional has certain "nice" conditional-like logical properties (such as *modus ponens*)
- The Field-conditional can be used to define a hierarchy of semantic deficiency predicates:

$$D(\phi) \equiv \phi \wedge (\top \mapsto \phi)$$

Vices

- the Field-conditional is complicated
- the Field-conditional has certain "unpleasant" un-conditional-like logical properties
  - $\not\models [\phi \wedge (\phi \mapsto \psi)] \mapsto \psi$
  - The theory is not closed under the semantic deduction rule

$$\frac{\phi \models \psi}{\phi \mapsto \psi}$$

- **the Field-conditional does not have a pure theoretical motivation**
  - interleaving of Kripkean and revision theoretic conditions

Kripke's inductive operator

Definition

Let *sk* be the Strong Kleene valuation scheme.

The Kripke jump operator I:

- $\phi \in \mathcal{I}_{\alpha+1}(T,+) \equiv \mathcal{I}_\alpha \models_{sk} \phi$
- $\phi \in \mathcal{I}_{\alpha+1}(T,-) \equiv \mathcal{I}_\alpha \models_{sk} \neg\phi$
- for λ limit: $\phi \in \mathcal{I}_\lambda(T,+) \equiv \phi \in \bigcup_{\alpha<\lambda}(\mathcal{I}_\alpha(T,+))$
- for λ limit: $\phi \in \mathcal{I}_\lambda(T,-) \equiv \phi \in \bigcup_{\alpha<\lambda}(\mathcal{I}_\alpha(T,-))$
- $\mathcal{I}_\alpha(\mapsto,+)$ and $\mathcal{I}_\alpha(\mapsto,+)$ are never revised by the operator I

(From now on we will drop the subscript "sk")

Kripke fixed points

Lemma
The Kripke jump operator is monotone.

Corollary
The Kripke jump has a least fixed point $\mathcal{I}_{lf}$.

Corollary
For every $\phi \in \mathcal{L}_T$:

$$\mathcal{I}_{lf} \models \phi \Leftrightarrow \phi \in \mathcal{I}_{lf}(T, +)$$

Yablo's operator

- $\phi \in Y_{\alpha+1}(T,+) \equiv$
  $\phi \in$ the extension of T of the least $I-$ fixed point extending Y_α
- $\phi \in Y_{\alpha+1}(T,-) \equiv$
  $\phi \in$ the anti-extension of T of the least $I-$ fixed point extending Y_α
- $\langle \phi, \psi \rangle \in Y_{\alpha+1}(\mapsto,+) \equiv$
  for every $\Psi \supseteq Y_\alpha$ and for all $I-$fixed points $\Theta \supseteq \Psi$:

  $$\Theta(\phi) \leq \Theta(\psi)$$

- $\langle \phi, \psi \rangle \in Y_{\alpha+1}(\mapsto,-) \equiv$
  for every $\Psi \supseteq Y_\alpha$ and for all $I-$fixed points $\Theta \supseteq \Psi$:

  $$\Theta(\phi) > \Theta(\psi)$$

- at limit stages, take unions

Virtues

- Yablo's construction is monotone and Kripkean ("theoretically pure")

Theorem

The operator Y as a least fixed point Y_{lf}.

- In Y_{lf} the Field-conditional behaves to some extent like a real conditional

Proposition

Y_{lf} validates Modus Ponens for $\mapsto$, $\frac{(A \mapsto B) \wedge (B \mapsto C)}{A \mapsto C}$,
$A \mapsto A, \ldots$

Vices

The account does terribly on conditionals that have other conditionals embedded within them.

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Proposition (Field)

The formula $(A \mapsto C) \mapsto ((A \vee A) \mapsto C)$ *is not valid in* Y_{If}.

Proof.

Take A, C *such that* $A \mapsto C$ *is gappy in* Y_{If}. *Then* Y_{If} *can be consistently extended with* $A \mapsto C$ *only and extended to an* I*-fixed point* Y^+. *In* Y^+, $A \mapsto C$ *will get the truth value 1, whilst* $((A \vee A) \mapsto C)$ *will be assigned the truth value* $\frac{1}{2}$. □

Yablo iteration?

One natural suggestion . . . would be to iterate Yablo's construction, so that the extensions of Y_{If} *are themselves Yablo fixed points.*

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> *One natural suggestion . . . would be to iterate Yablo's construction, so that the extensions of Y_{If} are themselves Yablo fixed points.*

Define Y^+ in terms of Y:

Definition

Let the operator Y^+ be defined exactly like the operator Y, except that "I-fixed point" is replaced everywhere by "Y-fixed point'.

Iteration blocked

Such hopes are dashed: the iterated version breaks down right from the start.

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`Proof` [Field]: use a Curry paradox argument.

Definition (Curry sentence)

$K \Leftrightarrow T(K) \mapsto \perp$

The Curry sentence K will have value $\frac{1}{2}$ in the stages of Y. So it will have value 0 in the first stage of Y^+. Thus we have that the least fixed point of Y_1^+, if there is one, is not even an Y-fixed point.

Pessimism

I take these to be clear deficiencies in Yablo's account as it now stands; there may conceivably be ways to fix the problems without altering its spirit, though my efforts in this regard haven't been successful.

A conditional jump operator 1

- $\phi \in \Phi^0_{\alpha+1}(T,+) \equiv \phi \in (I_{lf}(\Phi^0_\alpha))(T,+)$
- $\phi \in \Phi^0_{\alpha+1}(T,-) \equiv \phi \in (I_{lf}(\Phi^0_\alpha))(T,-)$
- $\langle \phi, \psi \rangle \in \Phi^0_{\alpha+1}(\mapsto,+) \equiv$ for all I-fixed points $\Theta \supseteq \Phi^0_\alpha : \Theta \models \phi \Rightarrow \Theta \models \psi$
- $\langle \phi, \psi \rangle \in \Phi^0_{\alpha+1}(\mapsto,-) \equiv$

$$\text{for all } I\text{-fixed points } \Theta \supseteq \Phi^0_\alpha : \Theta \models \phi \wedge \neg\psi$$

- at limit stages take unions

A conditional jump operator 2

Theorem

Φ^0 *has a least fixed point.*

For each such fixed point Φ^0_f of the inductive operator Φ^0:

- $\phi \in \Phi^0_f(T,+) \Leftrightarrow \Phi^0_f \models \phi$
- $\langle \phi, \psi \rangle \in \Phi^0_f(\mapsto,+) \Leftrightarrow$

$$\text{for all } I-\text{fixed points } \Theta \supseteq \Phi^0_f : \Theta \models \phi \Rightarrow \Theta \models \psi$$

- $\langle \phi, \psi \rangle \in \Phi^0_f(\mapsto,-) \Leftrightarrow$

$$\text{for all } I-\text{fixed points } \Theta \supseteq \Phi^0_f : \Theta \models \phi \wedge \neg\psi$$

Definition

The least fixed point of the operator Φ^0 when started on a structure S is denoted as $\Phi^0_{lf}(S)$.

Iteration

For each $n > 0$, the inductive operator Φ^n is defined as follows:

- $\phi \in \Phi^n_{\alpha+1}(T,+) \equiv \phi \in (\Phi^{n-1}_{lf}(\Phi^n_\alpha))(T,+)$
- $\phi \in \Phi^n_{\alpha+1}(T,-) \equiv \phi \in (\Phi^{n-1}_{lf}(\Phi^n_\alpha))(T,-)$
- $\langle \phi, \psi \rangle \in \Phi^n_{\alpha+1}(\mapsto,+) \equiv$ for all Φ^{n-1}-fixed points $\Theta \supseteq \Phi^n_\alpha$:
$$\Theta \models \phi \Rightarrow \Theta \models \psi$$
- $\langle \phi, \psi \rangle \in \Phi^n_{\alpha+1}(\mapsto,-) \equiv$
$$\text{for all } \Phi^{n-1}\text{-fixed points } \Theta \supseteq \Phi^n_\alpha : \Theta \models \phi \wedge \neg\psi$$
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Properties

Definition

The least fixed point of the operator Φ^n when started on a structure S is denoted as $\Phi^n_{lf}(S)$.

Theorem

Every Φ^n has a least fixed point.

Proposition

Every Φ^{n+1} fixed point is a Φ^n fixed point.

Embedded conditionals

`Observation:` The formula $(A \mapsto C) \mapsto ((A \vee A) \mapsto C)$ is in Φ^1_{lf}.

Thesis

Φ^n fixed points assign reasonable truth conditions to sentences with conditional nestings of depth at most $\leq n+1$.

Onwards and upwards

Definition

$$\Phi^{\omega}(S) \equiv \bigcup_{n} \Phi^{n}_{lf}(S)$$

Proposition

1. *Φ^{ω} has a least fixed point Φ^{ω}_{lf}*
2. *Φ^{ω}_{lf} is a fixed point of Φ^{n} for every n.*

Thesis

Φ^{ω}_{lf} assigns reasonable truth conditions to all sentences of $\mathcal{L}$ that have finite conditional nesting depth.

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Semantic deficiency

We can express indeterminacy in $\mathcal{L}$:

Definition

$$ID(\phi) \equiv (\phi \mapsto \bot) \wedge (\neg\phi \mapsto \bot)$$

- When $ID(\phi)$ is judged to be true by Φ^0_{If}, then ϕ meets an indeterminacy standard (Strong Kleene fixed point).
- If it judged to be true by Φ^n_{If} for some $n > 0$, then it meets a stricter indeterminacy condition.
- In a way that is familiar from the work of Field, we can diagonalise out of this indeterminacy predicate and thus generate a hierarchy of increasingly strong determinacy predicates.

Intrinsic truth

Definition

$$INT(\phi) \equiv \neg ID(\phi) \wedge [(\phi \vee \neg\phi) \mapsto \phi]$$

Proposition

$INT(\phi)$ holds in Φ^0_{lf} if and only if ϕ is intrinsically true.

- There are better notions of intrinsic-ness that can be defined in terms of Φ^n_{lf} for $n > 0$.
- Since the collection of intrinsic truths is complicated, this means that the collection of Φ^n_{lf}-truths must already be complicated.

Concluding thoughts

- Yablo's aspiration for a more "Kripkean" conditional is attractive.
- Yablo's idea can be pushed further.
- Semantic indeterminacy can be expressed in the resulting structures
- Intrinsic-ness can be expressed in the resulting structures

Concluding thoughts

- Yablo's aspiration for a more "Kripkean" conditional is attractive.
- Yablo's idea can be pushed further.
- Semantic indeterminacy can be expressed in the resulting structures
- Intrinsic-ness can be expressed in the resulting structures

Is the price not too high? Is the aspiration to "add a real conditional" to a theory of reflexive truth a reasonable one?

References

- Field, H. *Saving Truth From Paradox.* Oxford University Press, 2008.
- Kripke, S. *Outline of a theory of truth.* Journal of Philosophy, 1975.
- Yablo, S. *New grounds for naive truth theory.* In: J.C. Beall, *Liars and Heaps.* Oxford University Press, 2003.