Truth and Groundedness

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Content

Kripke on groundedness I

(1) Most (i.e, a majority) of Nixon's assertions about watergate are false.

In general, if a sentence such as (1) asserts that (all, some, most, etc.) of the sentences of a certain class C are true, its truth value can be ascertained if the truth values of the sentences in the class C are ascertained. If some of these sentences themselves involve the notion of truth, their truth value in turn must be ascertained by looking at other sentences, and so on. If ultimately this process terminates in sentences not mentioning the concept of truth, so that the truth value of the original statement can be ascertained, we call the original sentence grounded; otherwise, ungrounded. (Kripke [1975, p.693f.])

Kripke on groundedness II

Yablo identifies two aspects of groundedness:

- inheritance, bottom-up.
- dependence, top-down.

Now Kripke's Theory is very instructive about the inheritance aspect of semantic grounding, but it really does little to supply the dependence aspect of our intuition. (Yablo [1982, p.118])

Definition (Kripke)

A sentence φ is grounded iff $\#\varphi$ is in the extension or anti-extension of the minimal fixed-point.

The idea

- Semantical theories of truth usually focus on one fixed-point model (Minimal fixed-point, maximally intrinsic fixed-point)
- The structure of fixed-points is rich.
- Include all the fixed-points and its structure in one model, a possible worlds model.
- Every world should represent one possible fixed-point.
- Define φ is grounded as φ is true in all fixed-points or $\neg\varphi$ is.

Fixed-points

Definition (Visser)

Let $\mathbb{X} = (X, \leq)$ be a poset. Then $\mathbb{X}$ is a ccpo (coherent complete partial order) iff every consistent subset of X has a supremum, where $Y \subseteq X$ is consistent iff for all $x, y \in Y$, $\{x, y\}$ has an upper bound in $\mathbb{X}$.

Theorem (Visser's fixed point theorem)

Let $\mathbb{X} = (X, \leq)$ *be a ccpo and* Γ *a monotone operator on* $\mathbb{X}$, *then* $\mathbb{F}(\mathbb{X}, \Gamma) = (\{x \in X : x = \Gamma(x)\}, \leq\restriction \{x \in X : x = \Gamma(x)\})$ *is a ccpo.*

Possible worlds

- Language $\mathcal{L}^m = \mathcal{L}_A \cup \{T, N, P\}$.
- Frame $F = (W, R)$ and model $M = (F, f)$.
- W a set of possible worlds, R an accessibility relation and f a partial evaluation function.
- where each $w \in W$ is a partial model $(\mathcal{N}, (E_T, A_T), (E_N, A_N), (E_P, A_P))$.
- $f : W \to P(\omega)$.

Consistent extensions

- Let there be a fixed-Gödel coding of $\mathcal{L}^m$, such that $\#\varphi$ is the Gödelnumber of φ.
- $SENT := \{\#\varphi : \varphi \text{ is a sentence of } \mathcal{L}^m\}$.
- $NSENT := \omega \setminus SENT$.
- For $X \subseteq SENT$ we say that X is consistent iff $\#\varphi \in X \Rightarrow \#\neg\varphi \notin X$.
- $CONS := \{X \subseteq SENT : X \text{ is consistent }\}$.

Evaluation functions

Let $f : W \to CONS$.

Definition

Let $F = (W, R)$ be a frame and f an evaluation function.

$$E^w_T(f) := f(w) \qquad A^w_T(f) := NSENT \cup \{\#\varphi : \#\neg\varphi \in E^w_T(f)\}$$

$$E^w_N(f) := \bigcap^{v}_{wRv} E^v_T(f) \qquad A^w_N(f) := \bigcup^{v}_{wRv} A^v_T(f)$$

$$E^w_P(f) := \bigcup^{v}_{wRv} E^v_T(f) \qquad A^w_P(f) := \bigcap^{v}_{wRv} A^v_T(f)$$

Let EV_c be the set of consistent evaluation functions.
For $f, g \in EV_C$ define $f \leq g :\Leftrightarrow$ for all $w \in W, f(w) \subseteq g(w)$.
Let $\mathbb{E} := (EV_c, \leq)$.

Lemma

$\mathbb{E}$ *is a ccpo.*

$M, w \models^{SK} \varphi$

Definition

Let M be a model (F, f), with $F = (W, R)$ a frame and $f \in EV$.
$M, w \models^{SK} \varphi$ is defined as follows:

(i) $M, w \models^{SK} s = t$ iff $s^{\mathcal{N}} = t^{\mathcal{N}}$;

(ii) $M, w \models^{SK} \neg s = t$ iff $s^{\mathcal{N}} \neq t^{\mathcal{N}}$;

(iii) $M, w \models^{SK} Tt$ iff $t^{\mathcal{N}} \in E^{w}_{T}$;

(iv) $M, w \models^{SK} \neg Tt$ iff $t^{\mathcal{N}} \in A^{w}_{T}$;

(v) $M, w \models^{SK} Nt$ iff $t^{\mathcal{N}} \in E^{w}_{N}$;

(vi) $M, w \models^{SK} \neg Nt$ iff $t^{\mathcal{N}} \in A^{w}_{N}$;

(vii) $M, w \models^{SK} Pt$ iff $t^{\mathcal{N}} \in E^{w}_{P}$;

(viiii) $M, w \models^{SK} \neg Pt$ iff $t^{\mathcal{N}} \in A^{w}_{P}$;

$M, w \models^{SK} \varphi$

Definition (Cont.)

(ix) $M, w \models^{SK} \neg\neg\varphi$ iff $M, w \models^{SK} \varphi$;

(x) $M, w \models^{SK} \varphi \wedge \psi$ iff $M, w \models^{SK} \varphi$ and $M, w \models^{SK} \psi$;

(xi) $M, w \models^{SK} \neg(\varphi \wedge \psi)$ iff $M, w \models^{SK} \neg\varphi$ or $M, w \models^{SK} \neg\psi$;

(xii) $M, w \models^{SK} \forall x\varphi$ iff for all $n \in \omega$, $M, w \models^{SK} \varphi(\underline{n}/x)$;

(xiii) $M, w \models^{SK} \neg\forall x\varphi$ iff there exists a $n \in \omega$, $M, w \models^{SK} \neg\varphi(\underline{n}/x)$;

We also use $f, w \models^{SK} \varphi$ for $M, w \models^{SK} \varphi$ if $\mathfrak{M}$ is based on f and clear from the context.

Kripke Jump

Lemma

If f is consistent, then for all w, $\{\#\varphi : f, w \models^{SK} \varphi\}$ is again consistent.

The modal Kripke jump operator $\Delta : EV_c \to EV_c$ is defined as $\Delta(f)(w) := \{\#\varphi : f, w \models^{SK} \varphi\}$.

Lemma

Δ is a monotone operator on $\mathbb{E}$, i.e. $f \leq g \Rightarrow \Delta(f) \leq \Delta(g)$.

Closing off

Let F be a reflexive frame and f a consistent fixed-point. Then for all w the following principles hold, i.e. $f, w \models$

1. KF + consistency.
2. $\forall x(Snt(x) \to (N(x) \to T(x)))$
3. $\forall x(Snt(x) \to (T(\underset{\cdot}{N}(\dot{x})) \leftrightarrow N(x)))$
4. $\forall x(Snt(x) \to (T(\underset{\cdot}{\neg}\underset{\cdot}{N}(\dot{x})) \leftrightarrow P(\underset{\cdot}{\neg}x)))$
5. $\forall x(Snt(x) \to (T(\underset{\cdot}{P}(\dot{x})) \leftrightarrow P(x)))$
6. $\forall x(Snt(x) \to (T(\underset{\cdot}{\neg}\underset{\cdot}{P}(\dot{x})) \leftrightarrow N(\underset{\cdot}{\neg}x)))$

Soundness

Definition

Let $\mathbb{X}$ be a ccpo and Γ a monotone operator on $\mathbb{X}$ and $x \in X$.

- x is sound iff $x \leq \Gamma(x)$.
- $SOUND_\Gamma := \{x \in X : x \text{ is sound } \}$.

Lemma

If x is sound, then there is a fixed-point y, such that $x \leq y$.

The possible worlds of the model

Let Γ_{SK} be the simple Kripke jump operator for the language $\mathcal{L}_T$.
Let $\mathrm{SOUND}_{\Gamma_{SK}}$ be the set of sound and consistent sets of sentences.
Let W be a set of worlds indexed by the reals, i.e. $W := \{w_i : i \in \mathbb{R}\}$.

Definition

Let f be an evaluation function $f : W \to \mathrm{CONS}_{\mathcal{L}_m}$.
f is Γ_{SK}-complete iff for all $X \in \mathrm{SOUND}_{\Gamma_{SK}}$ there is a $w_i \in W$ such that $f(w_i) = X$.

h

Since $|\mathrm{SOUND}_{\Gamma_{SK}}| = 2^\omega$ there is a bijection $h : W \to \mathrm{SOUND}_{\Gamma_{SK}}$.
Clearly h is an evaluation function.

Lemma

- *h is a consistent evaluation function.*
- *h is sound with respect to Δ_{SK}.*
- *h is Γ_{SK}-complete.*

The model

Let $F = (W, R)$ be a frame, with W as before and R universal, i.e. for all $i, j \in \mathbb{R}, w_i R w_j$.
We construct a fixed-point model $\mathfrak{M}$ based on h. Let $h_0 = h$, then:

$$\mathfrak{M}_0 = (W, R, h_0)$$
$$\mathfrak{M}_{\alpha+1} = (W, R, \Delta_{SK}(h_\alpha))$$
$$\mathfrak{M}_\beta = (W, R, \bigcup_{\alpha<\beta} h_\alpha).$$

Let w_0 be the world such that $h(w_0) = \emptyset$.
We will be mainly interested in sentences true in the model $\mathfrak{M}$ at w_0.

Groundedness defined

Definition

$$G(x) :\leftrightarrow N(x) \vee N(\underset{\cdot}{\neg}\, x)$$

Examples

$\mathfrak{M}, w_0 \models$

$$\begin{aligned} G(\ulcorner 2+2=4 \urcorner) \\ G(\ulcorner 2+2=3 \urcorner) \\ G(\ulcorner T \ulcorner 2+2=4 \urcorner \urcorner) \\ G(\ulcorner T \ulcorner 2+2=3 \urcorner \urcorner) \end{aligned}$$

More general:

$$\mathfrak{M}, w_0 \models \forall x(Snt_{\mathcal{L}_A}(x) \rightarrow G(x)).$$

Liars and Truth tellers

More examples: Let

$$\lambda \leftrightarrow \neg T(\ulcorner \lambda \urcorner) \qquad \text{liar}$$
$$\tau \leftrightarrow T(\ulcorner \tau \urcorner) \qquad \text{truthteller}$$

Then

$$\mathfrak{M}, w_0 \models \neg G(\ulcorner \lambda \urcorner)$$
$$\mathfrak{M}, w_0 \models \neg G(\ulcorner \tau \urcorner)$$

Groundedness and truth

$\mathfrak{M}, w_0 \models$

1. $G(x) \rightarrow Snt(x)$.
2. $Snt_{\mathcal{L}_A}(x) \rightarrow G(x)$.
3. $G(x) \rightarrow T(x) \vee T(\underset{\cdot}{\neg} x)$.
4. $G(x) \rightarrow (T(\underset{\cdot}{\neg} x) \leftrightarrow \neg T(x))$.
5. $G(\ulcorner \varphi \urcorner) \rightarrow (T(\ulcorner \varphi \urcorner) \leftrightarrow \varphi)$, for all $\varphi \in \mathcal{L}^m$.

Groundedness and compositionality

$\mathfrak{M}, w_0 \models$

1. $G(x) \leftrightarrow G(\underset{\cdot}{\neg}\, x)$.
2. $G(x) \wedge G(y) \rightarrow G(x \underset{\cdot}{\wedge} y) \wedge G(x \underset{\cdot}{\vee} y)$.
3. $G(x \underset{\cdot}{\wedge} y) \vee G(x \underset{\cdot}{\vee} y) \rightarrow G(x) \vee G(y)$.
4. $\forall y G(x(\dot{y})) \rightarrow G(\underset{\cdot}{\forall}\, z x(z))$.
5. $G(x) \leftrightarrow G(\underset{\cdot}{T}\, \dot{x})$.

Groundedness and necessity

What about the following:

- $G(x) \leftrightarrow G(\underset{\cdot}{N}(\dot{x}))$
- $G(x) \leftrightarrow G(\underset{\cdot}{P}(\dot{x}))$

Example:

$$G(\underset{\cdot}{N}(\dot{x})) \leftrightarrow N(\underset{\cdot}{N}(\dot{x})) \vee N(\underset{\cdot}{\neg}\underset{\cdot}{N}(\dot{x})) \leftrightarrow N(x) \vee P(\underset{\cdot}{\neg}x).$$

Only in special models where the extension of the necessity predicate is the same as the extension of the possibility predicate. For example in the case where only one world is considered and necessity and possibility are reduced to truth.

Excurs: weak Kleene

- The strong Kleene and weak Kleene scheme are compositional in contrast to supervaluation.
- In terms of determinateness, $D(x) \leftrightarrow T(x) \vee T(\dot{\neg} x)$:
  - supervaluation: $D(x \vee y) \not\rightarrow D(x) \vee D(y)$. Example: $\lambda \vee \neg\lambda$.
  - strong Kleene: $D(x \vee y) \rightarrow D(x) \vee D(y)$ but not $D(x \vee y) \rightarrow D(x) \wedge D(y)$, for example $\lambda \vee \ulcorner 2 + 2 = 4 \urcorner$.
  - weak Kleene: $D(x \vee y) \rightarrow D(x) \wedge D(y)$.

Determinateness and Groundedness

For weak Kleene we could also have the converse direction of 2 and 3, if we make sure that we evaluate T in a fixed-point, such that all other fixed-points that are reachable contain the extension.

Lemma

Let $\mathfrak{M}, w$ be a closed-off model for weak Kleene and R such that, if wRv, then $E_T^w \subseteq E_T^v$. Then for all $w \in W$, $\mathfrak{M}, w \models G(x) \wedge G(y) \leftrightarrow G(x \dot{\wedge} y)$ and $\mathfrak{M}, w \models G(x) \wedge G(y) \leftrightarrow G(x \dot{\vee} y)$.

Theorem

There is a model based on weak Kleene (modified) that models DT.

Feferman's DT

(i) $AtSnt_{\mathcal{L}_A}(x) \to D(x)$.

(ii) $D(T(\dot{x})) \leftrightarrow D(x)$.

(iii) $Snt_{\mathcal{L}}(x) \to (D(x) \leftrightarrow D(\underset{\cdot}{\neg} x))$

(iv) $Snt_{\mathcal{L}}(x) \wedge Snt_{\mathcal{L}}(y) \to (D(x \underset{\cdot}{\vee} y) \leftrightarrow D(x) \wedge D(y))$

(v) $Snt_{\mathcal{L}}(x) \wedge Snt_{\mathcal{L}}(y) \to (D(x \underset{\cdot}{\to} y) \leftrightarrow D(x) \wedge (T(x) \to D(y)))$

(vi) $Var(z) \wedge Snt_{\mathcal{L}}(\underset{\cdot}{\forall} z\, x) \to (D(\underset{\cdot}{\forall} z\, x) \leftrightarrow \forall y D(x(z/\dot{y})))$

(vii) $Ct(s) \wedge Ct(t) \to (T(s \underset{\cdot}{=} t) \leftrightarrow val(s) = val(t))$.

(viii) $D(x) \to (T(T(\dot{x})) \leftrightarrow T(x))$.

(ix) $Snt_{\mathcal{L}}(x) \wedge D(x) \to (T(\underset{\cdot}{\neg} x) \leftrightarrow \neg T(x))$

(x) $Snt_{\mathcal{L}}(x) \wedge Snt_{\mathcal{L}}(y) \wedge D(x \underset{\cdot}{\vee} y) \to (T(x \underset{\cdot}{\vee} y) \leftrightarrow T(x) \vee T(y))$

(xi) $Snt_{\mathcal{L}}(x) \wedge Snt_{\mathcal{L}}(y) \wedge D(x \underset{\cdot}{\to} y) \to (T(x \underset{\cdot}{\to} y) \leftrightarrow (T(x) \to T(y)))$

(xii) $Var(z) \wedge Snt_{\mathcal{L}}(\underset{\cdot}{\forall} z\, x) \wedge D(\underset{\cdot}{\forall} z\, x) \to (T(\underset{\cdot}{\forall} z\, x) \leftrightarrow \forall y T(x(z/\dot{y})))$

Paradoxicality and intrinsicality

Besides the notion of groundedness we can define:

Definition

- $paradoxical(x) :\leftrightarrow \neg P(x) \wedge \neg P(\underset{\cdot}{\neg}\, x)$
- $intrinsicallytrue(x) :\leftrightarrow P(x) \wedge \neg P(\underset{\cdot}{\neg}\, x)$
- $intrinsicallyfalse(x) :\leftrightarrow P(\underset{\cdot}{\neg}\, x) \wedge \neg P(x)$

Examples:

- $\mathfrak{M}, w_0 \models paradoxical(\lambda)$
- $\mathfrak{M}, w_0 \models \neg paradoxical(\tau)$
- For $\pi \leftrightarrow T(\ulcorner \pi \urcorner) \vee T(\ulcorner \neg\pi \urcorner)$:
  $\mathfrak{M}, w_0 \models intrinsicallytrue(\pi)$

Inner and outer logic

Problem: negatively formulated predicates:

- $u \leftrightarrow \neg G(\ulcorner u \urcorner)$
- $p \leftrightarrow paradoxical(\ulcorner p \urcorner)$

Then

- $\mathfrak{M}, w_0 \models u$.
- $\mathfrak{M}, w_0 \models p$.
- $\mathfrak{M}, w_0 \models \neg T(\ulcorner u \urcorner)$.
- $\mathfrak{M}, w_0 \models \neg T(\ulcorner p \urcorner)$.

Neither u, p are grounded.

Necessarily true versus true in the minimal fixed-point

Let E^I be the extension of the minimal fixed-point of the original strong-Kleene jump Γ_{SK}.
Kripke: Grounded (φ) iff $\#\varphi \in E^I$ or $\#\neg\varphi \in E^I$.
Two questions:

1. $\mathfrak{M}, w_0 \models \forall x(Sent(x) \to (G(x) \leftrightarrow T(x) \vee T(\underset{\cdot}{\neg}x)))$?
2. For all $\varphi \in \mathcal{L}_T$: Grounded (φ) iff $\mathfrak{M}, w_0 \models G(\ulcorner\varphi\urcorner)$?

Necessarily true versus true in the minimal fixed-point

$\mathfrak{M}, w_0 \models \forall x(Sent(x) \to (G(x) \leftrightarrow T(x) \vee T(\underset{.}{\neg} x)))$.

Reasons for preferring necessarily true:

- We are not forced to choose w_0 as our preferred model in order to get an adequate extension of the groundedness predicate.
  - We could also choose for example the maximally intrinsic fixed-point w_i to evaluate the truth predicate.
  - If we choose a universal accessibility relation the extension of the groundedness predicate is the same for all worlds.
- We could use the model to define relative notions of groundedness.

Relative notions of groundedness

- Let Σ be in $\mathrm{SOUND}_{\Gamma_{SK}}$.
- Let w_Σ be the world associated with Σ, i.e. $h(w_\Sigma) = \Sigma$.
- Let $R_\leq$ be the accessibility relation with wRv iff $h(w) \subseteq h(v)$.
- Let $\mathfrak{M}_\leq$ be the fixed-point model based on $(W, R_\leq)$.

Definition

φ is grounded in Σ iff $\mathfrak{M}_\leq, w_\Sigma \models G(\ulcorner \varphi \urcorner)$

Indirect uses of possibility

- For all $\varphi \in \mathcal{L}_T$: If Grounded (φ), then $\mathfrak{M}, w_0 \models G(\ulcorner \varphi \urcorner)$.
- The other direction does not hold: The reason being indirect uses of the new predicates inside the truth predicate.
- For example: Choose a truth teller τ such that $\mathrm{PAT} \vdash \tau \leftrightarrow T(\ulcorner P(\ulcorner \tau \urcorner) \urcorner) \vee T(\ulcorner \tau \urcorner)$. Then τ is in $\mathcal{L}_T$. Moreover τ is sound with respect to Γ_{SK} and therefore will be in the extension of some fixed-point. So $P(\ulcorner \tau \urcorner)$ will be true at all w and also τ. Therefore $G(\ulcorner \tau \urcorner)$ will be true at w_0 but τ is neither in the extension nor anti-extension of the minimal fixed-point.

Indirect uses: possible solutions

- Change the accessibility relation: $R \Rightarrow R_{\leq}$.
  Then $P(\ulcorner\tau\urcorner)$ will not be true at all worlds.
  Problem: not a general solution! Consider
  $\tau \leftrightarrow T(\ulcorner P(\ulcorner\tau\urcorner)\urcorner) \vee T(\ulcorner P(\ulcorner\neg\tau\urcorner)\urcorner) \vee T(\ulcorner\tau\urcorner)$
- Exclude the indirect uses:
  Define $\mathcal{L}_T^-$ to be the language $\mathcal{L}$ without implicit uses of N, P, such that for all $\varphi \in \mathcal{L}_T^-$: Grounded (φ) iff $\mathfrak{M}, w_0 \models G(\ulcorner\varphi\urcorner)$
  Problem: Is there an adequate restriction?

Universal languages and the revenge problem

- The present model is not supposed to be a step in the direction of a model for a universal language.
- The present model is not supposed to solve the revenge problem.

Open questions

- Supervaluation models?
- Adequate axiomatizations?
- Partial logic? Modalized PKF?
- Groundedness as a primitive notion?
- Analogous dependency models?

Conclusion

The model

- is adequate with respect to a large part of the language.
- is very flexible
  - different evaluation schemes.
  - different accessibility relations.
- is a model of an expressive theory of truth and groundedness.
- gives some insight into the interaction of a truth and a groundedness predicate.

Thank you!