Adding standardness to nonstandard models

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If M is a nonstandard model of PA, the set of standard natural numbers $\omega = \{0, 1, 2, \ldots\}$ forms an initial segment of M in a canonical way. It therefore makes sense to look at the model (M, ω) formed by adding a predicate for ω to M. Model theory of (M, ω) is rather hard. Such structures encode a number of very difficult questions in second order number theory (analysis), including many open ones. However some attempt can be made to understand (M, ω) relative to second order schemes of arithmetic.

The talk will touch on three related questions: what can one interpret or define in (M, ω) that cannot be defined in M; what is the theory of structures such as (M, ω); and what reals are coded in the model (M, ω)?

In terms of definability and interpretability, structures (M, ω) interpret ω-models of second order arithmetic, and also under certain circumstances define truth predicates for submodels of M. Thus second order systems motivate this work on structures (M, ω). It turns out that the truth predicate has useful applications and some details of how it arises will be given.

In terms of the theory, the Henkin-Orey theorem on ω-logic tells us about the theory of all models (M, ω) (i.e. the statements true in all such models) but tells us little about the theory of any specific model. In fact the theory of (M, ω) depends on structural properties of M that are not first order, and so there is a wide range of possibilities for the theory of (M, ω), even for a given completion T of PA. It is perhaps surprising therefore that given a complete theory T extending PA there is a canonical choice for a theory Th(M, ω) of some $M \models T$. More surprisingly, this is not hard to prove. We will discuss a few consequences of this result, and further applications of the truth predicates available in (M, ω) will be given.

In a difficult paper in the JSL, Kanovei characterised the Scott sets Rep(M, ω) of subsets of ω that are 0-definable in (M, ω), when M is a model of true arithmetic. A similar characterisation of the standard system SSy(M, ω) of (M, ω) (i.e. such definable sets, where parameters are allowed from M) is not known. We will conclude with some results and observations on these standard systems, with some open problems for future work.

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