Revision without ordinals

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Roughly speaking, semantic theories of Truth deal with the problem of constructing a model for a given language together with an interpretation of the Truth-predicate for this language. To set things in a more suitable way for a mathematical treatment we can assume — as it is often the case in the literature on this topic — that (1) a formalized first-order version of Peano Arithmetic (PA) plays the role of the object language, (2) a self-referential Truth-predicate T is added to the language $\mathcal{L}$ of PA and applies to sentences of the extended language $\mathcal{L} + \{\mathsf{T}\}$ and (3) the interpretation of T over the standard model of PA has to be built up within Zermelo-Fraenkel (ZF) set theory (or within some weaker fragment of ZF).

Several semantic theories of self-referential Truth can be presented as systems of ordinal-length sequences of attempts to give an interpretation for the Truth-predicate. This strategy provides a general mathematical framework in which different approaches to Truth can be fruitfully compared: for an analysis of this kind see Barba [1] where both constructions based on *approximation sequences* (as in Kripke [6]) and constructions based on *revision sequences* (Gupta and Belnap [4] is the standard reference) are covered.

While some knowledge of natural numbers seems to be unavoidable in inquiring about Truth — since, at least, we need a theory of the syntax of the truth-bearers (sentences, in our setting) and this theory comes to be equivalent to some weak form of arithmetic — the introduction of heavy set-theoretical assumptions in the metatheory could be regarded as an unwelcome accident in a theory of Truth.

As a matter of fact, the approximation constructions can also be presented in an inductive style, without any reference to the ordinals, so reducing the amount of set-theoretical notions needed in the metalanguage (see Fitting [3] for a detailed treatment). Apparently, this possibility constitutes a substantive difference between the approximation and the revision approaches: but it will be shown that this is not the case.

From a result by McGee [8] we already know that, in inquiries about Truth for countable languages, we can restrict ourselves to countable revision sequences. This fact suggests that it might be possible to do the Revision Theory without ordinals just by reducing ordinals to natural numbers via some index notation for countable ordinals.

I will present here a different approach to this problem that directly shows that the mathematical features of revision sequences that are relevant for the Revision Theory of Truth can be implemented in an ordinal-free setting — worked out in the Dedekind-Kuratowski tradition — as in the approximation case.

This strategy of doing revision without ordinals can show its merits under several aspects:

- General motivations for eliminating the ordinals from some pieces of mathematics can be found in Kuratowski [7]:

  > "Even though, sometimes, transfinite numbers [ordinals] can be shown to be fruitful in making the exposition shorter or easier, the existence of a process that allows to avoid ordinals, in proving theorems that do not deal with the transfinite, is important for the following two reasons: in reasoning about ordinals we implicitly appeal to axioms that ensure their existence; but to weak the axioms system that we use in

proving something is desirable both from a logical and from a mathematical point of view. Moreover, this strategy expunges from the arguments the unnecessary elements, increasing their aesthetic value.[4]"

- Our ordinal-free presentation of the Revision Theory of Truth, together with Fitting's ordinal-free exposition of Kripke's theory, provides a framework in which to compare approximation and revision approaches to Truth that constitutes an alternative to Barba's analysis based on ordinal-length sequences.
- An ordinal-free presentation makes easier to evaluate how the Revision Theory of Truth is sensitive to its underlying set theory: both in order to minimise the set-theoretical assumptions taken from the Zermelo-Fraenkel axiomatization and in order to explore the possibility of doing revision also in alternative set theories.
- Avoiding ordinals may help in facing the problem of lifting the revision-theoretical approach from "toy" object languages like arithmetic to more complex languages, since the entire set-theoretical process of revision can be recast in higher-order logical terms.
- The ordinal-free approach to the Revision Theory of Truth might be shown valuable also from a heuristic point of view, helping to focus our attention on the intrinsic circularities exhibited by the concept of Truth rather than on the redundancies introduced in the analysis by the transfinite iteration of the revision operator.

I plan to illustrate some of the above points as follows. First, I will briefly recall an overview of the Revision Theory of Truth. Then, I will introduce the mathematical notion of *generalized orbit* and its basic properties. Finally, we discuss how this latter concept can be successful in replacing the notion of *revision sequence* as the fundamental notion for a theory of Truth.

Bibliography

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[4] A. Gupta, N. Belnap: *The revision theory of truth*, 1993

[5] H. G. Herzberger: *Notes on Naive Semantics*, Journal of Philosophical Logic 11 n. 1: 61-102, 1982

[6] S. Kripke: *Outline of a theory of truth*, Journal of Philosophy 72: 690-716, 1975

[7] C. Kuratowski: *Une méthode d'élimination de nombres transfinis des raisonnements mathémathiques*, Fundamenta Mathematicae 3: 76-108, 1922

[8] V. McGee: *Truth, Vagueness and Paradox. An essay on the logic of truth*, 1991

[4]"Bien que l'emploi des nombres transfinis puisse présenter quelquefois quelques avantages au point de vue de la brièveté et de la simplicité, l'existence d'un procédé permettant de supprimer la notion de ces nombres dans les démonstrations des théorèmes qui ne concernent guère le transfini est importante pour les deux raisons suivantes: en raisonnant avec le nombres transfinis on fait implicitement usage de l'axiome de leur existence; or, la réduction du système d'axiomes employés dans les démonstrations est désirable au point de vue logique et mathématiques. En outre, cette réduction affranchit les raisonnements de l'élément qui leur est étranger, ce qui augment leur valeur esthétique" (the English translation is mine).