The disentanglement of syntax from a model theoretic point of view

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In the field of formal theories of truth a prominent place is occupied by the (broadly) Tarskian theory. The theory is also called "there is a (full-not inductive) satisfaction class" or, shortly, PA$(S)^-$. It consists, apart from the axioms of the base theory PA, of the truth-compositional axioms (here I consider arithmetical induction only):

1. $\forall\varphi(\text{Atomic}(\varphi) \to (\text{Tr}(\varphi) \leftrightarrow \text{Tr}^*(\varphi)$;
2. $\forall\varphi(\text{Tr}(\neg\varphi) \leftrightarrow \neg\text{Tr}(\varphi)$;
3. $\forall\varphi\forall\psi(\text{Tr}(\varphi \wedge \psi) \leftrightarrow \text{Tr}(\varphi) \wedge \text{Tr}(\psi))$;
4. $\forall\varphi\forall i(\text{Tr}(\forall v_i\varphi) \leftrightarrow \forall t\text{Tr}(\varphi(t/v_i))$;[5]

The name PA$(S)^-$ should be explained, because it carries important information and leads to debatable aspects.

When studying truth theories, it is often said that a background theory of syntax is needed. Without it, writing down axioms for a truth predicate and working out simple operations is impossible. We intend to ascribe truth to so called "truth-bearers"; a theory of syntax is intended to give us basic information about how these entities behave. One would expect a theory of syntax to consist of principles about linguistic expressions, and this was exactly the case in the original work of Tarski. However, explicit formal theories of syntax, in the style of concatenation theories i.e., are not much widespread among truth-theorists.

The reason is that, after Gödel, we know that a very good deal of syntax can be developed inside PA (as in even weaker arithmetical theories). According to Gödelization, we can correlate natural numbers and symbols of the language of PA. There are many ways to think of this correspondence between strings of symbols and numbers, but one often adopted is the easiest one: strings of symbols are identified with corresponding numbers.

Since PA is a very well known theory and we often want to consider the effects of adding a theory of truth to a theory of arithmetic like PA, it is clear that PA is the best candidate for a theory of syntax in many cases. Indeed, we can add our theory of truth directly to PA, without describing an independent theory of syntax. All of this sounds completely familiar and unproblematic, but it happens that a lot of intricacies survive here.

Among the many syntactical properties that can be represented in PA, we can obviously define that of being a sentence of the language of PA. For example there is a formula 'Sent(n)' which is true of n if and only if n is the code of a sentence of the language of PA. Until we consider the standard model N, as

[5] This way of writing is comfortable but incorrect. A lot of coding apparatus has been suppressed to achieve a greater readability. To be rigorous we should write axiom 2, for example, like this: $\forall x\forall y(\text{Sent}(x) \wedge \text{Sent}(y) \wedge \text{Neg}(y,x) \to (\text{Tr}(y) \leftrightarrow \neg\text{Tr}(x)))$. Here I shall persist with the most perspicuous presentation, but keep in mind that this is the right form.

it is natural doing, this works as expected. PA, however, also has non-standard models. In those non-standard models, because of Overspill, the formula 'Sent(x)' is going to be satisfied by non-standard numbers too. Namely, if $M \models$ PA is non-standard, we have that $M \models$ Sent(b) for some $b \in M$, and b non-standard. The existence of non-standard numbers that, according to the model, code "sentences", drags us towards the realm of non-standard sentences. Very roughly, non-standard sentences are sentences with a "non-standard structure".[6]

Since every axiom of PA$(S)^-$ is subjected to a clause stating that the truth predicate applies to elements satisfying the formula 'Sent(x)', when we have a non-standard model, non-standard numbers can well enter into the range of the truth predicate. Actually this is not only possible but mandatory. In fact, PA$(S)^-$ proves $\forall\varphi\{\text{Sent}(\varphi) \rightarrow [\text{Tr}(\varphi) \vee \neg\text{Tr}(\varphi)]\}$, thus, for every φ such that $M \models \text{Sent}([\varphi])$ either φ or $\neg\varphi$ must be in the extension of "Tr", even if φ is non-standard.

Now let S be this extension. Namely, given a model $M \models$ PA, S is the set of numbers satisfying the axioms of PA$(S)^-$, $M, S \models$ PA$(S)^-$. When a model M has such a set S, we say that S is a satisfaction class for M. This explains the name PA$(S)^-$.

One of the most notable results about PA$(S)^-$ is the Lachlan's theorem, according to which not every model of PA has a satisfaction class. This result becomes even more surprising confronted with the fact that PA$(S)^-$ is (proof theoretically) conservative over PA. Thus, that some models are excluded is not due to new theorems in the language of PA.

Because of conservativity, one may well find — or, at least, I really found it so — the fact that in some model M we cannot have a suitable extension for the truth predicate to be a puzzling result. Actually, it really seems that something, somewhere, is gone wrong. Ingenuously, it seems that a set S of sentences satisfying the axioms of PA$(S)^-$ is always available, actually. Aren't we entitled to do model theory in every model of PA? After all, a set of sentences satisfying PA$(S)^-$ exists in every M, as "$M \models$" proves. But, according to Lachlan's theorem, we cannot exploit this fact. What is crucial is the existence, in non-standard models, of non-standard sentences, which make Lachlan's proof work. The problem is not only technical, it is a general philosophical one, since we do not recognize non-standard sentences as real sentences and we shouldn't be forced to apply our theory of truth to them, despite their mathematically interesting nature.

The source of the phenomenon is not hard to identify. It lays in the fact that, when writing down the Tarskian axioms for truth, we quantify in the object language, the language of PA, through the formula 'Sent(x)'. Doing so, anytime we have a non-standard model M of the base theory, we are forced to consider non-standard elements satisfying it, obtaining non-standard sentences. This is symptomatic of the fact that we developed our syntactic theory in the base theory. We have one single theory doing two different jobs. On the one side we treat it as our base arithmetical theory, on the other side as our syntactical theory. Having one single theory simplifies the issue, but forces the two functions to overlap and intertwine. This gives unpleasant results: in one precise sense, the base language has no non-standard sentences, and we would like to have a theory of syntax[7] not forced to them only because the base theory has non-standard models. In the present framework, instead, non-standard numbers automatically become non-standard sentences. If the base theory is about non-standard objects, we must conclude that we have non-standard sentences in the object language.

Clearly, we are not forced to take seriously non-standard sentences when we specify the semantics for the language of PA in general. Non-standard sentences are a side-effect of our formalization. Their existence shows that such a formalization makes incomplete justice to our meta-mathematical reasoning about the semantics of the language of PA.

The natural solution to avoid these kinds of side-effects is that of restoring the original Tarskian

[6] For example, a sentence with a non-standard number of conjuncts

[7] Clearly, a theory of syntax may also have non standard models. The problem, though, is that here the models of the base theory and those of the syntax theory are the same models, so that what happens at the level of arithmetical objects, also happens at level of syntactical objects.

attitude, namely distinguishing neatly the theory of syntax from the base theory. For sake of brevity, here I give just a quick sketch to get a rough idea of how we can proceed. We start with a theory of syntax S in the language L_S.[8] Since we want to devise syntax and semantics for a further language, say L_{PA}, we also need to be able to refer to the expressions of L_{PA}, and the objects L_{PA} is about. The simplest way is then to adopt the language $L_S \cup L_{\text{PA}}$. Some precaution should be certainly adopted here, in order to take the intended domains apart. At this point we can add semantic axioms, namely axioms for truth and denotation.[9] Call the complex theory of truth (for L_{PA}) we obtain "Σ".

It can be proved that every model M of PA can be expanded to a model of Σ. The exact proof will depend on the technical details, that, of course, must be spelled out; what is important, however, is the simple idea of this proof. According to it, this time, we can restrict our attention to standard sentences: we can interpret, by simple stipulation, the expressions of the object language in the standard part of M. Disentangling the theory of syntax from the base theory, then, gives us what we wanted: we are not forced to consider non-standard sentences in our truth theory, just because the base theory has non-standard models. Being able to deal with standard sentences only allows us to circumvent Lachlan's theorem, since a set S, satisfying compositional axioms, is available in any model of PA; in particular, it is a set of standard sentences, as expected.

The idea of a disentanglement of the theory of syntax from the base theory has been (re)proposed very recently by Richard Heck, who argued from mere proof-theoretic considerations. Basically, he stressed that, using a truth theory, we can, unpleasantly, prove consistency statements for PA already in weaker fragments of PA. This can be done - cutting a very and interesting story short - only because we go freely back and forth from syntax to numbers, inside the same theory. If we disentangle syntax, it becomes clear that those consistency statements belong to the theory of syntax only, and the unfortunate results are overcome.

Volker Halbach stressed that there is something puzzling in this approach as well. As a matter of fact we know that the truth of consistency statements, formulated in a metalanguage, corresponds to the truth of certain statements in the object language. This aspect is lost in Heck's framework. If we want to make justice to our meta-mathematical reasoning and formalize it adequately, then, a formal reconstruction of Gödelization, and bridge laws connecting the two sides, must be added.

I see the simple model theoretic considerations above as a different way to arrive at the same attitude of Heck and Halbach. In particular, I take these reflections as another good reason for disentangling syntax and, at the same time, as a confirm of Halbach's diagnosis: the identification between numbers and expressions is certainly legitimate and mathematically highly useful, but it should handled with extraordinary care.

Bibliography

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[3] Halbach, V. *Axiomatic Theories of Truth*, Cambridge University Press, Cambridge, 2011.

[8] We may also keep using PA as our syntax theory, provided that we label and use it as a different theory from the arithmetical base theory.

[9] The addition of separate axioms for denotation, or for sequences in case we have a satisfaction predicate, is a little tricky. Notice however, that no trick is needed until we do not disentangle the syntax from the base theory. In particular, we have variables, which are syntactical objects, already identified with numbers, so a bridge theory connecting the two domains is not needed.

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