Transplendent models

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Preliminaries

- All models will be (expansions of) models of PA.
- All languages $\mathscr{L}$ will be recursive extensions of the language of arithmetic, $\mathscr{L}_A$.
- The standard system of M, $\mathrm{SSy}(M)$, is the collection of standard parts of (parameter) $\mathscr{L}_A$-definable sets in M.

$$\mathrm{SSy}(M) = \{\, X \cap \mathbb{N} \mid X \in \mathrm{Def}(M') \,\},$$

where M' is the $\mathscr{L}_A$-reduct of M.

Recursive saturation

- A type over M is a set of formulas with finitely many parameters $\bar{a}$ from M and finitely many free variables $\bar{x}$ consistent with $\mathrm{Th}(M, \bar{a})$.
- M is recursively saturated if all recursive types over M are realized in M.
- M is recursively saturated iff all types in $\mathrm{SSy}(M)$ are realized in M.
- There is a Σ^1_1-sentence Θ such that a model is recursively saturated iff it satisfies Θ.
- Θ says that M-logic is consistent.

Resplendent models

- M is resplendent if for any theory T in an expanded language $\mathscr{L} \supseteq \mathscr{L}_A \cup \{\bar{a}\}$ such that $T + \mathrm{Th}(M, \bar{a})$ is consistent there is an expansion of M satisfying T.
- All resplendent models are recursively saturated.
- All countable recursively saturated models are resplendent.
- There is a Δ^1_2 sentence Θ such that a model is resplendent iff it satisfies Θ.
- Θ says that M-logic is consistent and that for every (non-standard) sentence φ consistent in M-logic there is a satisfaction class including φ.

Subresplendent models

- M is subresplendent if for any theory T in an expanded language $\mathscr{L} \supseteq \mathscr{L}_A \cup \{\bar{a}\}$ such that $T + \mathrm{Th}(M, \bar{a})$ is consistent there are an elementary submodel $\bar{a} \in N$ of M and an expansion of N satisfying T.
- A model is subresplendent iff it is recursively saturated.

Arithmetic saturation

- M is arithmetically saturated if for any type arithmetic in some $\mathrm{Th}(M, \bar{a})$, $\bar{a} \in M$, is realized in M.
- M is arithmetically saturated iff M is recursively saturated and $\mathrm{SSy}(M)$ is closed under arithmetic comprehension.
- A countable recursively saturated model M is arithmetically saturated iff there is a maximal automorphism, i.e. an automorphism f such that

$$\mathrm{fix}(f) = \{ a \in M \mid f(a) = a \} = \mathrm{Scl}_M(\emptyset).$$

Omitting types

- $p{\uparrow}$ is the $\mathscr{L}_{\omega_1\omega}$-sentence saying that p is omitted, i.e.,

$$\forall \bar{x} \bigvee_{\psi(\bar{x}) \in p(\bar{x})} \neg\psi(\bar{x})$$

- $p(x)$ is isolated in T if there is $\varphi(x)$ such that $T + \exists x \varphi(x)$ is consistent and $T \models \forall x \big(\varphi(x) \rightarrow \psi(x)\big)$ for all $\psi(x) \in p(x)$.
- Omitting Types Theorem: If $p(x)$ is not isolated in T then $T + p{\uparrow}$ is consistent.

Omitting types, two examples

- If M is countable and recursively saturated and $p(x)$ is a type which is not isolated in any $T + \mathrm{Th}(M, \bar{a}, \bar{b})$ where $\bar{b} \in M$ (and these theories are consistent) then there is an expansion of M satisfying $T + p\uparrow$.
- Maximal automorphism: Let T_f say that f is an automorphism and $p_f(x) = \{f(x) = x\} \cup \{\, x \neq t \mid t \text{ is a Skolem term}\,\}$.
- f is a maximal automorphism of M iff $(M, f) \models T_f + p_f\uparrow$.
- p_f is isolated in $\mathrm{Th}(M, a)$, where $a \notin \mathrm{Scl}_M(\emptyset)$.
- Standard cut: Let $T_K = \{\, K(n) \mid n \in \mathbb{N}\,\}$ and $p_K(x) = \{K(x)\} \cup \{\, x > n \mid n \in \mathbb{N}\,\}$.
- K is the standard cut of M iff $(M, K) \models T_K + p_K\uparrow$.
- p_K is isolated in $\mathrm{Th}(M, a)$, where a is non-standard.

Definition

- T and $p(x)$ are a theory and a type in an extended language $\mathscr{L}$ with finitely many parameters $\bar{a}$ from M. T_0 a $\mathscr{L}_A$ theory.
- $T + p\!\uparrow$ is fully consistent over T_0 if there is a model of $T_0 + T + p\!\uparrow$ with standard system $\mathscr{P}(\mathbb{N})$ whose $\mathscr{L}_A$-reduct is recursively saturated.
- $T + p\!\uparrow$ is fully consistent over M if it is fully consistent over $\mathrm{Th}(M, \bar{a})$.
- Let $M \models \mathrm{PA}$. We say that M is almost transplendent if for all $T, p(\bar{x}) \in \mathrm{SSy}(M)$ such that $T + p\!\uparrow$ is fully consistent over M, there is an expansion M^+ of $(M, \bar{a})$ such that $M^+ \models T + p\!\uparrow$ and $\mathrm{Th}(M^+, \bar{a}) + p\!\uparrow$ is fully consistent over M.~~and $\mathrm{Th}(M^+, \bar{a}) + p\!\uparrow$ is fully consistent over M.~~

Existence

Theorem

There is a transplendent model of cardinality $2^{\aleph_0}$.

Definition

A Scott set $\mathscr{X}$ is *closed* if for any $T_0, T, p \in \mathscr{X}$ such that $T + p\uparrow$ is fully consistent over T_0 there is a completion $T_c \in \mathscr{X}$ of T such that $T_c + p\uparrow$ is fully consistent over T_0.

Theorem

Any countable recursively saturated model with a closed standard system is transplendent.

Closure is equivalent to transplendence

Given $T, p \in \mathrm{SSy}(M)$ such that $T + p{\uparrow}$ is fully consistent over T_0 then the theory

$$T + p{\uparrow} + \text{`}c \text{ codes the theory of the model'}$$

is fully consistent over T_0.

Theorem

Let M be a countable model of PA. M is transplendent iff M is recursively saturated and $\mathrm{SSy}(M)$ is closed.

The standard predicate

- Let $T_{K=\mathbb{N}}$ be the theory consisting of all $K(n)$, $n \in \mathbb{N}$ together with $p\!\uparrow$ where $p(x) = \{K(x)\} \cup \{\, x > n \mid n \in \mathbb{N}\,\}$.
- $(M, K) \models T_{K=\mathbb{N}}$ iff $K = \mathbb{N}$.
- Clearly $T_{K=\mathbb{N}}$ is fully consistent over any T_0.

Second order arithmetic vs. the standard predicate

Given a second order $\mathscr{L}_A$-sentence Θ we translate it into a first order $\mathscr{L}_A(K)$-sentence Θ^K:

- $(t \in X)^K$ becomes $(x)_t \neq 0$
- $(\exists x \Psi)^K$ becomes $\exists x (K(x) \wedge \Psi^K)$
- $(\exists X \Psi)^K$ becomes $\exists x \Psi^K$

Theorem

$(M, \mathbb{N}) \models \Theta^K(\bar{a})$ *iff* $\mathrm{SSy}(M) \models \Theta(\bar{A})$, *where* $\bar{a}$ *codes the sets* $\bar{A}$.

Implications of transplendence

If $\mathscr{P}(\mathbb{N}) \models \Theta(\bar{A})$ then $T_{K=\mathbb{N}} + \Theta^K(\bar{a})$ is fully consistent over any T_0.

Theorem

If M is transplendent then $\mathrm{SSy}(M) \prec \mathscr{P}(\mathbb{N})$.

Given $A \in SSy(M)$ then $T_{K=\mathbb{N}} +$ 'c codes $\mathrm{Th}(\mathscr{P}(\mathbb{N}), A)$' is fully consistent over any T_0.

Theorem

If M is transplendent then $\mathrm{Th}(\mathscr{P}(\mathbb{N}), A) \in \mathrm{SSy}(M)$ for every $A \in \mathrm{SSy}(M)$.

Some basis theorems

- $\Gamma \subseteq \mathscr{P}(\mathbb{N})$ is a basis for $\Delta \subseteq \mathscr{P}(\mathscr{P}(\mathbb{N}))$ if $\forall X{\in}\Delta\ X \cap \Gamma \neq \emptyset$.
- $V = L$ implies that Δ^1_k is a basis for Σ^1_k for all $k \geq 2$.
- PD implies that Δ^1_k is a basis for Σ^1_k for all *even* $k \geq 2$.
- If $V = L$ or PD then Σ^1_∞ is a basis for Σ^1_∞.

Theorem

If $V = L$ or PD then if $\mathrm{Th}(\mathscr{P}(\mathbb{N}), A) \in \mathscr{X}$ for every $A \in \mathscr{X}$ then $\mathscr{X} \prec \mathscr{P}(\mathbb{N})$.

Is this true in general?

Weakening 'fully consistent'

Replacing 'fully consistent' with 'consistent' in the definition of transplendent models will not work: Let $p(x)$ be some type realized in M.

Theorem

There are recursive T, p such that

- *For any type q over PA there is a model of $PA + T + q{\downarrow} + p{\uparrow}$.*
- *No recursively saturated model of PA has an expansion satisfying $T + p{\uparrow}$.*

$T + p{\uparrow}$ is
$T_{K=\mathbb{N}}$ + 'Σ is a truth predicate' + 'there is an omitted coded type'.

Subtransplendent models

What can be done if 'fully consistent' is weakened to 'consistent'?

Definition

We say that M is *subtransplendent* if for all $T, p(\bar{x}) \in \mathrm{SSy}(M)$ such that $T + p{\uparrow} + \mathrm{Th}(M, \bar{a})$ is consistent there are an elementary submodel $(N, \bar{a}) \prec (M, \bar{a})$ and an expansion $N^+ \models T + p{\uparrow}$ of $(N, \bar{a})$.

Theorem

M is subtransplendent iff M is recursively saturated and for every $T, p \in \mathrm{SSy}(M)$ such that $T + p{\uparrow} + \mathrm{Th}(M, \bar{a})$ is consistent there is a completion $T_c \in \mathrm{SSy}(M)$ of T making $T_c + p{\uparrow} + \mathrm{Th}(M, \bar{a})$ consistent.

β-models

Definition

$\mathscr{X} \subseteq \mathscr{P}(\mathbb{N})$ is a β-model if $\mathscr{X} \prec_{\Sigma^1_1} \mathscr{P}(\mathbb{N})$.

Theorem

M is subtransplendent iff M is recursively saturated and $SSy(M)$ is a β-model.

Transplendence implies subtransplendence.

The logic of omitting a type

- A Scott set $\mathscr{X}$ is a β-model iff for every $T_0, T, p \in \mathscr{X}$ such that $T + p{\uparrow} + T_0$ is consistent there is a completion $T_c \in \mathscr{X}$ of T making $T_c + p{\uparrow} + T_0$ consistent.
- For every hyperarithmetic T and p the height of a (minimal) proof in the logic of omitting a type (a variant on ω-logic) is at most ω_1^{CK}.
- There are hyperaritmetic T and p such that the supremum of the heights of (minimal) proofs is ω_1^{CK}.
- The supremum of all heights of (minimal) proofs over all recursive T, p is at least ϵ_0.

Open questions

- Is it possible to replace fully consistency with something weaker?
- Almost transplendence implies transplendence?
- Nicer characterization of the standard systems of transplendent models.
- What's the complexity of the notion of transplendence/subtransplendence?
- Is there a satisfaction class type property such that M is transplendent iff there is such a satisfaction class?