A NOTE ON STANDARD SYSTEMS AND ULTRAFILTERS

FREDRIK ENGSTRÖM

Abstract. Let $(M, \mathscr{X}) \models \mathrm{ACA}_0$ be such that $P_{\mathscr{X}}$, the collection of all unbounded sets in $\mathscr{X}$, admits a definable complete ultrafilter and let T be a theory extending first order arithmetic coded in $\mathscr{X}$ such that M thinks T is consistent. We prove that there is an end-extension $N \models T$ of M such that the subsets of M coded in N are precisely those in $\mathscr{X}$. As a special case we get that any Scott set with a definable ultrafilter coding a consistent theory T extending first order arithmetic is the standard system of a recursively saturated model of T.

The standard system of a model M of PA (the first order formulation of Peano arithmetic) is the collection of standard parts of the parameter definable subsets of M, i.e., sets of the form $X \cap \omega$, where X is a parameter definable set of M, and ω is the set of natural numbers. It turns out that the standard system tells you a lot about the model; for example, any two countable recursively saturated models of the same completion of PA with the same standard system are isomorphic. A natural question to ask is then which collections of subsets of the natural numbers are standard systems. This problem has become known as the *Scott set problem*.

For countable models the standard systems are exactly the countable Scott sets, i.e., countable boolean algebras of sets of natural numbers closed under relative recursion and a weak form of König's lemma [Scott, 1962]. It follows, by a union of chains argument (see [Knight and Nadel, 1982]), that for models of cardinality at most $\aleph_1$ the standard systems are exactly the Scott sets of cardinality at most $\aleph_1$.

If the continuum hypothesis holds this settles the Scott set problem. However, if CH fails then very little is known about standard systems of models of cardinality strictly greater than $\aleph_1$, although it is easy to see that any standard system of any model is a Scott set (see [Kaye, 1991]).

The key notion of a definable ultrafilter plays a central role in this paper. Suppose $\mathscr{X}$ is a collection of sets of natural numbers, and $P_{\mathscr{X}}$ is the collection of infinite members of $\mathscr{X}$. A filter F on $P_{\mathscr{X}}$ is said to be definable if for all $A \in \mathscr{X}$ $\{\, a \in \omega \mid (A)_a \in F \,\} \in \mathscr{X}$, where $(A)_a = \{\, b \in \omega \mid \langle a, b\rangle \in A \,\}$ and $\langle \cdot, \cdot \rangle$ is some canonical pairing function. The central theorem of this paper is Theorem 1 below (whose proof will be presented in section 2).

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THEOREM 1. *Let $\mathscr{X}$ be a Scott set that carries a definable ultrafilter and let $T \in \mathscr{X}$ be a consistent completion of PA. Then there is a recursively saturated model of T with standard system $\mathscr{X}$.*

It should be noted that all Scott sets that carry a definable ultrafilter are arithmetically closed (see Lemma 6) and that any countable arithmetically closed Scott set carries a definable ultrafilter, thus our result partly extends Scott's theorem (Theorem 2 below).

Our hope is that this paper may shed some light on the Scott set problem for uncountable standard systems. We will end the paper by offering some hints as to how this might be done.

Some of the results in this paper can also be found in the author's doctoral thesis [Engström, 2004].

§1. Background and preliminaries. Let $\mathscr{L}_A$ be the language of arithmetic with symbols $0, 1, +, \cdot, <$ and PA the ordinary axiomatization of arithmetic in first order logic. Let $\mathbb{N}$ be the standard model of PA and ω be its domain, i.e., the set of natural numbers. If $a \in M \models \mathrm{PA}$ let $I^M_{<a}$ (or $I_{<a}$ if M is understood from the context) be the initial segment $\{ b \in M \mid M \models b < a \}$ of M. $I \subseteq_e M$ means that I is an initial segment of M, i.e., $I^M_{<a} \subseteq I$ for all $a \in I$. $I \subseteq_e M$ is a cut if it is closed under taking successors. Let $x \in y$ be a Δ_0-formula in the language of arithmetic used for coding sets in $\mathrm{I}\Sigma_1$ (see [Kaye, 1991]).[1] If I is a cut in M and $a \in M$ then $\mathrm{set}_{M/I}(a) = \{ b \in I \mid M \models b \in a \}$. The set of all coded subsets of I will be denoted $\mathrm{Cod}(M/I) = \{ \mathrm{set}_{M/I}(a) \mid a \in M \}$. The standard system of M is $\mathrm{SSy}(M) = \mathrm{Cod}(M/\omega)$.

If $I \subseteq M$ is a cut closed under multiplication and $\mathscr{X} \subseteq \mathscr{P}(I)$ we can regard the structure $(I, \mathscr{X})$ as a second order structure in the language $\mathscr{L}_A$. The subtheories RCA_0, WKL_0 and ACA_0 of second order arithmetic are as defined in [Simpson, 1999]. A cut $I \subseteq M$ is semiregular iff $(I, \mathrm{Cod}(M/I)) \models \mathrm{WKL}_0$, and strong iff $(I, \mathrm{Cod}(M/I)) \models \mathrm{ACA}_0$.

When we write $(M, \mathscr{X})$ it will be understood that $(M, \mathscr{X})$ is a second order structure in the language $\mathscr{L}_A$, i.e., that M is equipped with addition and multiplication and $\mathscr{X} \subseteq \mathscr{P}(M)$. $(M, \mathscr{X})$ is an ω-model if $M = \mathbb{N}$ is the standard model of PA. In this case $(\mathbb{N}, \mathscr{X}) \models \mathrm{WKL}_0$ iff $\mathscr{X}$ is a Scott set. We will often specify an ω-model by only specifying $\mathscr{X} \subseteq \mathscr{P}(\omega)$.

Given a theory T, in the language of arithmetic, we say that a set $A \subseteq \omega$ is represented in T if there is a formula $\varphi(x)$ such that $T \vdash \varphi(n)$ for all $n \in A$ and $T \vdash \neg\varphi(n)$ for all $n \in \omega \setminus A$. By $\mathrm{rep}(T)$ we denote the collection of all sets represented in T. In [Scott, 1962] Scott proved a variant of the following theorem:

THEOREM 2. *For any countable ω-model $\mathscr{X}$ and complete consistent theory $T \supseteq \mathrm{PA}$ the following are equivalent:*

(i) $\mathscr{X} \models \mathrm{WKL}_0$ *and* $\mathrm{rep}(T) \subseteq \mathscr{X}$, *and*

(ii) there is a countable nonstandard model $M \models T$ *with* $\mathrm{SSy}(M) = \mathscr{X}$.

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[1] $\mathrm{I}\Sigma_1$ is PA with the induction axioms restricted to Σ_1 formulas.

This theorem could be taken a bit further: Given a Scott set $\mathscr{X}$ of cardinality $\aleph_1$ there is a nonstandard model of PA with standard system $\mathscr{X}$: [2]

THEOREM 3. *For any ω-model $\mathscr{X}$ of cardinality at most $\aleph_1$ and complete consistent theory $T \supseteq \mathrm{PA}$ the following are equivalent:*

(*i*) $\mathscr{X} \models \mathrm{WKL}_0$ *and* $\mathrm{rep}(T) \subseteq \mathscr{X}$, *and*
(*ii*) *there is a nonstandard model* $M \models T$ *with* $\mathrm{SSy}(M) = \mathscr{X}$.

The proof is based on a union of chains argument. It should be noted that for Presburger arithmetic, $\mathrm{PR} = \mathrm{Th}(\omega, +)$, this argument could be extended to any cardinality $\leq 2^{\aleph_0}$ as proved in [Knight and Nadel, 1982].[3] For PA the argument only applies for cardinalities less than or equal $\aleph_1$, since models of PA does not admit the amalgamation property needed, as pointed out in [Knight and Nadel, 1982].

For recursively saturated models the situation is almost the same as Wilmers proved in [Wilmers, 1975]:

THEOREM 4. *For any ω-model $\mathscr{X}$ and complete consistent theory $T \supseteq \mathrm{PA}$ the following are equivalent:*

(*i*) $T \in \mathscr{X}$ *and there is a model* $M \models T$ *with* $\mathrm{SSy}(M) = \mathscr{X}$, *and*
(*ii*) *there is a recursively saturated model* $M \models T$ *with* $\mathrm{SSy}(M) = \mathscr{X}$.

The proof is a rather straightforward application of the arithmetized completeness theorem.

Observe that if $\mathscr{X}$ is a Scott set and $T \in \mathscr{X}$ then $\mathrm{rep}(T) \subseteq \mathscr{X}$, however it may well happen that $\mathrm{rep}(T) \subseteq \mathscr{X}$ and $T \notin \mathscr{X}$.

Given $(M, \mathscr{X})$ and $I \subseteq M$ a cut, we say that $A \subseteq I$ is coded in $\mathscr{X}$ if there is $B \in \mathscr{X}$ such that $A = B \cap I$. Also, if $A \subseteq I \times J$ where J also is a cut we say that A is coded in $\mathscr{X}$ if there is $B \in \mathscr{X}$ such that $B \cap I \times J = A$.

Given a theory T let Con_T be the first order theory consisting of the sentences Con_S, where $S \subseteq T$ ranges over all finite subtheories of T and Con_S is the sentence saying that $\neg\sigma$ is not provable in first order logic, where σ is the conjunction of all sentences in S.

The next theorem is a special version of the arithmetized completeness theorem tailormade for our purposes.

THEOREM 5. *Let $(M, \mathscr{X}) \models \mathrm{ACA}_0 + \mathrm{Con}_T$, where T is some first order theory extending* PA *coded in $\mathscr{X}$. Then there is a model $N \models T$ with domain M and a set $C \in \mathscr{X}$ such that $N \models \varphi(a)$ iff $\langle \varphi, a \rangle \in C$. Furthermore, the canonical embedding $\varrho : M \to N$ is coded in $\mathscr{X}$ and such that $\varrho(M)$ is an initial segment of N.*

[2] In [Smoryński, 1984], Smorýnski gives Ehrenfeucht and Jensen [Ehrenfeucht and Jensen, 1976] and independently Guaspari [Guaspari, 1979] the honor of the main lemma used to prove the theorem. He also writes that the observation that the theorem follows from the lemma is due to Guaspari in [Guaspari, 1979]. It is not clear to us if this is correct. What is clear is that the lemma and the theorem appears explicitly in Knight and Nadel [Knight and Nadel, 1982].

[3] For models of PR we have to define the standard system to be the collection of sets recursive in a complete type realized in the model, this definition is equivalent to our definition for recursively saturated models of PA.

PROOF. Let $A \in \mathscr{X}$ code the theory T, i.e., $T = A \cap \omega$. By the assumption on $(M, \mathscr{X})$ we have $(M, \mathscr{X}) \models \mathrm{Con}_{x \in A \wedge x < n}$ for every $n \in \omega$, where $\mathrm{Con}_{\theta(x)}$ is the sentence saying that there is no proof of absurdity from sentences satisfying the formula $\theta(x)$.

If $M = \mathbb{N}$ then clearly $(M, \mathscr{X}) \models \mathrm{Con}_{x \in A}$ by the compactness theorem. Assuming M is nonstandard we can use overspill (this is where we need $(M, \mathscr{X})$ to satisfy ACA_0) to find $T \subseteq B \in \mathscr{X}$ such that $(M, \mathscr{X}) \models \mathrm{Con}_{x \in B}$.

Using the fact that the completeness theorem is provable in WKL_0 we get the desired set C (see [Simpson, 1999]). ⊣

Let $(P, <)$ be a partial order. A filter $F \neq P$ on P is an upwards closed nonempty subset of P such that if $x, y \in F$ then there is $z \in F$ satisfying $z \leq x$ and $z \leq y$. A filter F is an ultrafilter if it is a maximal filter.

Given $(M, \mathscr{X})$ let $P_{\mathscr{X}}$ be the partial order of all unbounded sets in $\mathscr{X}$ ordered by $\subseteq$. A filter F on $P_{\mathscr{X}}$ is complete if for all $f : M \to M$ coded in $\mathscr{X}$ with a bounded range there is $A \in F$ such that f is constant on A. Any complete filter on $P_{\mathscr{X}}$ is a nonprincipal ultrafilter on the boolean algebra $\mathscr{X}$. Also, if $\mathscr{X}$ is an ω-model any nonprincipal ultrafilter on $\mathscr{X}$ is a complete filter on $P_{\mathscr{X}}$.

Recall that a filter F on $P_{\mathscr{X}}$ is definable if for all $A \in \mathscr{X}$ $\{ a \in M \mid (A)_a \in U \} \in \mathscr{X}$, where $(A)_a = \{ b \in M \mid \langle a, b \rangle \in A \}$ and $\langle \cdot, \cdot \rangle$ is some canonical function coding pairs. We will, somewhat sloppily, say that a filter U on $P_{\mathscr{X}}$ is an ultrafilter if U is an ultrafilter on $\mathscr{X}$.

LEMMA 6 ([Kirby, 1984]). *If* $(M, \mathscr{X}) \models \mathrm{RCA}_0$ *and there is a definable ultrafilter* U *on* $P_{\mathscr{X}}$ *then* $(M, \mathscr{X}) \models \mathrm{ACA}_0$.

PROOF. To see this let $B = \{ a \in M \mid M \models \exists x \varphi(a, x, A) \}$, where $A \in \mathscr{X}$ and φ is Δ^0_0. Define $C = \{ \langle a, b \rangle \mid \exists x{<}b\ \varphi(a, x, A) \} \in \mathscr{X}$, then

$$(C)_a = \{ b \in M \mid \exists x{<}b\ \varphi(a, x, A) \}$$

and thus $B = \{ a \in M \mid (C)_a \in U \} \in \mathscr{X}$ since U is definable. ⊣

§2. The construction. We are now in a position to start proving the following theorem. Theorem 1 will follow from it.

THEOREM 7. *If* $(M, \mathscr{X}) \models \mathrm{ACA}_0$, $P_{\mathscr{X}}$ *has a definable complete filter* F *and* $T \supseteq \mathrm{PA}$ *is coded in* $\mathscr{X}$ *such that* $M \models \mathrm{Con}_T$, *then there is an end-extension* $N \models T$ *of* M *satisfying* $\mathrm{Cod}(N/M) = \mathscr{X}$.

First we construct the ultrapower we will use to build the model N.

Given the setup of Theorem 7 let K_0 be the model of T given by Theorem 5 and let $\varrho : K \cong K_0$ be such that $M \subseteq_e K$ and $\varrho \restriction M$ is the canonical embedding of M into K_0.

Let $\prod_{\mathscr{X}} K$ be the set of all functions $f : M \to K$ such that the function $\varrho \circ f : M \to M$ is coded in $\mathscr{X}$.

For any ultrafilter U on $P_{\mathscr{X}}$ define $\prod_{\mathscr{X}} K/U$ to be the set of equivalence classes of the equivalence relation $\equiv_U$ defined on $\prod_{\mathscr{X}} K$ by $f \equiv_U g$ iff the set where f and g are equal, $\{ a \in M \mid f(a) = g(a) \}$, is in U. The collection $\prod_{\mathscr{X}} K/U$ of equivalence classes can be interpreted as a structure in the language of arithmetic by the ordinary definitions of functions and relations. Let N denote some model of the form $\prod_{\mathscr{X}} K/U$, where U will be understood to be an ultrafilter on $P_{\mathscr{X}}$.

Let σ be a sentence in the language $\mathscr{L}_A(\prod_{\mathscr{X}} K)$, i.e. the language of arithmetic extended with the set $\prod_{\mathscr{X}} K$ as parameters. The $\mathscr{L}_A(K)$-sentence we get by replacing all occurrences of functions f by the value $f(i)$ will be denoted by $\sigma[i]$. By $[\sigma]$ we mean the $\mathscr{L}_A(N)$-sentence we get by replacing all functions f by the equivalence class $[f]$.

The Łoś theorem in this setting follows:

Lemma 8. *For any sentence σ of $\mathscr{L}_A(\prod_{\mathscr{X}} K)$ we have $N \models [\sigma]$ iff*

$$\{\, i \in M \mid K \models \sigma[i] \,\} \in U.$$

Let us embed K in N via the canonical embedding $F : K \to N$, $F(a) = [f_a]$, where $f_a(b) = a$ for all $b \in M$. We will identify $a \in K$ with $F(a) \in N$ making K a substructure of N. In fact Łoś theorem gives us that that $K \prec N$ and thus that $N \models T$.

Let us summarize. We have $M \subseteq_e K \prec N$, where $K \models T$ and there is a model K_0 with the same domain as M such that $\mathrm{Th}(K_0, a)_{a \in K_0}$ is coded in $\mathscr{X}$ and $\varrho : K \to K_0$ is an isomorphism.

Lemma 9. *If U is complete then $M \subseteq_e N$.*

Proof. Let $[f] \in N$ be such that $N \models [f] < a$, $a \in M$. Take $g \in [f]$ such that $g(b) < a$ for all $b \in M$. Observe that the function $\varrho \restriction M : M \to M$ is coded in $\mathscr{X}$ and that $g(M) \subseteq M$. Since $g(M)$ is bounded in M so is $\varrho(g(M))$.

Now $\varrho \circ g : M \to M$ has bounded range and thus, by the completeness of U, there is $A \in U$ such that $\varrho(g(A)) = \{\, b \,\}$, $b \in M$. Therefore $N \models [f] = [g] = \varrho^{-1}(b)$ and thus $[f] \in M$. ⊣

Lemma 10. *For any complete ultrafilter U on $P_{\mathscr{X}}$, $\mathscr{X} \subseteq \mathrm{Cod}(N/M)$.*

Proof. Given $A \in \mathscr{X}$ we will find $[f] \in N$ such that $[f]$ codes A, i.e., $N \models a \in [f]$ iff $a \in A$, for all $a \in M$.

Let $f(a)$ be the least code in M of the bounded set $A \cap I_{<a}$. Since $M \subseteq_e K$ we have $M \prec_{\Delta_0} K$ and thus $f(a)$ also codes the set $A \cap I_{<a}$ in K.

It should be clear that f is coded in $\mathscr{X}$. For any $a, b \in M$ we have $K \models a \in f(b)$ iff $a < b$ and $a \in A$, and so

$$\begin{aligned} a \in A \quad &\Rightarrow \quad \{\, b \in I \mid K \models a \in f(b) \,\} = I \setminus I_{<a+1}, \\ a \notin A \quad &\Rightarrow \quad \{\, b \in I \mid K \models a \in f(b) \,\} = \emptyset. \end{aligned}$$

Thus, $a \in A$ iff $\{\, b \in M \mid K \models a \in f(b) \,\} \in U$, i.e., iff $N \models a \in [f]$. ⊣

Lemma 11. *If U is complete and definable then $\mathscr{X} \supseteq \mathrm{Cod}(N/M)$.*

Proof. We show that $\mathrm{set}_{N/M}([f]) \in \mathscr{X}$ for any $[f] \in N$. By Łoś's theorem

$$\mathrm{set}_{N/M}([f]) = \{\, a \in M \mid \{\, b \in M \mid K \models a \in f(b) \,\} \in U \,\}.$$

Thus, if $A = \{\, \langle a, b \rangle \mid K \models a \in f(b) \text{ and } a, b \in M \,\} \in \mathscr{X}$ then $\mathrm{set}_{N/M}([f]) = \{\, a \in M \mid (A)_a \in U \,\} \in \mathscr{X}$ by the definability of U. ⊣

Proof of Theorem 7. Given $(M, \mathscr{X}) \models \mathrm{ACA}_0$ and U a definable and complete ultrafilter on $P_{\mathscr{X}}$ and a theory $T \supseteq \mathrm{PA}$ coded in $\mathscr{X}$ such that $M \models \mathrm{Con}_T$, let N be $\prod_{\mathscr{X}} K/U$ where K is as above. By the preceding lemmas $N \models T$, $M \subseteq_e N$ and $\mathrm{Cod}(N/M) = \mathscr{X}$. ⊣

Now we are ready to prove the central theorem of this paper, Theorem 1. Let us recall it.

Theorem 1. *Let $\mathscr{X}$ be a Scott set that carries a definable ultrafilter and let $T \in \mathscr{X}$ be a consistent completion of PA. Then there is a recursively saturated model of T with standard system $\mathscr{X}$.*

Proof. If $\mathscr{X}$ is a Scott set admitting a definable ultrafilter then $(\mathbb{N}, \mathscr{X}) \models \mathrm{ACA}_0$. Any ultrafilter on $\mathscr{X}$ is complete so Theorem 7 gives us a model $N \models T$ with $\mathrm{SSy}(N) = \mathscr{X}$. An application of Theorem 4 gives a recursively saturated model of T with standard system $\mathscr{X}$. ⊣

A natural question to ask is whatever every $\mathrm{Cod}(M/I)$ where I is a strong cut in M admits a complete definable nonprincipal ultrafilter. This is not the case; Enayat, in [Enayat, 2006b], constructed a Scott set $\mathscr{X} \models \mathrm{ACA}_0$ of cardinality $\aleph_1$ with no nonprincipal definable ultrafilter. However, by Theorem 3 and 4 there is a recursively saturated $M \models \mathrm{PA}$ such that $\mathrm{SSy}(M) = \mathscr{X}$. That ω is strong in M follows from the fact that $\mathscr{X} \models \mathrm{ACA}_0$.

§3. Definability and forcing. Theorem 1 partly reduces the question of constructing models of arithmetic with standard system $\mathscr{X}$ to the question when we can find definable ultrafilters on $P_{\mathscr{X}}$. One way of constructing these ultrafilters is by finding generic ultrafilters.

Definition 12. Let P be a partial order and $(M, \mathscr{X}) \models \mathrm{WKL}_0$.

- A set $D \subseteq P$ is dense in P if for every $x \in P$ there is $y \in D$ such that $y \leq x$.
- If $\mathscr{D}$ is a set of dense sets in P a filter F is *$\mathscr{D}$-generic* if $F \cap D \neq \emptyset$ for every $D \in \mathscr{D}$.
- A filter on $P_{\mathscr{X}}$ is generic if it is $\mathscr{D}$-generic, where $\mathscr{D}$ is the collection of all dense sets (parameter) definable in $(M, \mathscr{X})$.

It is well known that any generic filter is definable, but see [Enayat, 2006a] for a proof.

Lemma 13. *If $(M, \mathscr{X}) \models \mathrm{ACA}_0$ and U is a generic ultrafilter on $P_{\mathscr{X}}$ then U is complete and definable.*

A partial order $(P, <)$ is said to have the countable chain condition (c.c.c. for short) if for every uncountable set $A \subseteq P$ there are $x, y \in A$ such that x and y are compatible, i.e., there is $z \in P$ such that $z < x$ and $z < y$.

Martin's axiom, MA for short, says that for any partial order P with the c.c.c. and any collection $\mathscr{D}$ of dense sets of cardinality $< 2^{\aleph_0}$ there is a $\mathscr{D}$-generic filter on P. Clearly, $\mathrm{ZFC} + \mathrm{CH} \vdash \mathrm{MA}$, but it is also the case that if ZFC is consistent then so is $\mathrm{ZFC} + \mathrm{MA} + \neg\mathrm{CH}$. In fact, if ZFC is consistent and $\kappa \geq \omega_1$ is regular such that $2^{<\kappa} = \kappa$, then $\mathrm{ZFC} + \mathrm{MA} + 2^{\aleph_0} = \kappa$ is consistent.

We can now state the following promising looking corollary:

COROLLARY 14. *If* MA *holds,* $|\mathscr{X}| < 2^{\aleph_0}$ *is an arithmetically closed Scott set, i.e.,* $\mathscr{X} \models \mathrm{ACA}_0$*, such that* $P_{\mathscr{X}}$ *has the c.c.c. and* $T \supseteq \mathrm{PA}$ *is a consistent theory coded in* $\mathscr{X}$ *then there is a recursively saturated* $N \models T$ *such that* $\mathrm{SSy}(N) = \mathscr{X}$*.*

However, as Hamkins and Gitman recently proved in [Hamkins and Gitman, 2005], there are no uncountable such sets $\mathscr{X}$.

THEOREM 15. *If* $(M, \mathscr{X}) \models \mathrm{RCA}_0$ *is such that* $P_{\mathscr{X}}$ *has the c.c.c. then* $\mathscr{X}$ *is countable.*

PROOF. For every $A \subseteq M$, let A^* be the set of codes of initial segments of A, i.e., $A^* = \{ \langle A \cap I_{<a} \rangle \mid a \in M \}$, where $\langle B \rangle$ is the least code in M of the bounded set B. It should be clear that if $X \in \mathscr{X}$ then $X^* \in \mathscr{X}$.

Also if $X \neq Y \in P_{\mathscr{X}}$ then $X^* \cap Y^*$ is bounded and thus X^* and Y^* are incompatible elements in $P_{\mathscr{X}}$. Therefore, the set $\{ X^* \mid X \in P_{\mathscr{X}} \}$ is an anti-chain in $P_{\mathscr{X}}$ of the same cardinality as $\mathscr{X}$. ⊣

Thus, this construction together with Martin's axiom does not give us anything new on the existence of standard systems.

There might still be some hope of at least proving the consistency of ZFC + ¬CH + '$\mathrm{ACA}_0 \subseteq \mathrm{SSy}$', where '$\mathrm{ACA}_0 \subseteq \mathrm{SSy}$' denotes the sentence saying that every arithmetically closed Scott set is the standard system of some model of PA. This might be done by forcing generic ultrafilters into a model of ZFC + ¬CH. However, for this mission to succeed we have to control the cardinals while doing the forcing. The following result of Enayat [Enayat, 2006b] is somewhat discouraging in that direction.

THEOREM 16. *There is a Scott set* $\mathscr{X} \models \mathrm{ACA}_0$ *such that forcing with* $P_{\mathscr{X}}$ *collapses* $\aleph_1$*.*

Recently Gitman [Gitman, 2006] used similar ideas to show that assuming the proper forcing axiom (PFA) any proper arithmetically closed Scott set is a standard system. Gitman also proved the consistency of ¬CH together with the existence of an arithmetically closed proper Scott sets of size $\aleph_1$. However it is, to our knowledge, open if PFA is consistent with the existence of such a Scott set.

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DEPARTMENT OF PHILOSOPHY
UNIVERSITY OF GOTHENBURG
BOX 200, 405 30 GÖTEBORG, SWEDEN
E-mail: fredrik.engstrom@filosofi.gu.se