Multivalued dependencies and generalized quantifiers

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Introduction

Branching

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Mulivalued Dependencies

Branching

Motivation from natural languages

Some relative of each villager and some relative of each townsmen hate each other. (Hintikka 1974)

$$\begin{pmatrix} \forall x \exists y \\ \forall z \exists w \end{pmatrix} \big(V(x) \wedge T(z) \rightarrow (R(x, y) \wedge R(z, w) \wedge H(y, w)) \big)$$

Most of the dots and most of the stars are all connected by lines. (Barwise 1979)

Each of two examiners marked each of six scripts. (Davies 1989)

Branching

For monotone quantifiers the branching of Q_1 and Q_2

$$\begin{pmatrix} Q_1 x \\ Q_2 y \end{pmatrix} R(x, y)$$

is $\mathrm{Br}(Q_1, Q_2)xy\,R(x, y)$, where $\mathrm{Br}(Q_1, Q_2)$ is the quantifier

$$\{ R \mid \exists A \in Q_1, B \in Q_2, A \times B \subseteq R \} .$$

Example:

$$R \in \mathrm{Br}(\forall\exists, \forall\exists)$$

iff

$$\exists\, S_1, S_2 \in \forall\exists \text{ such that } S_1 \times S_2 \subseteq R$$

iff

$$\exists f, g : M \to M \text{ such that } \forall x, z\, R\big(x, f(x), z, g(z)\big)$$

Branching in Dependence Logic

$$M \vDash \mathrm{Br}(\forall\exists, \forall\exists)xyzw\, R(x, y, z, w)$$

iff

$$M \vDash \forall x \exists y \forall z \exists w \left(=\!(z, w) \wedge R(x, y, z, w)\right)$$

What about generalized quantifiers?

$$M \vDash \mathrm{Br}(Q_1, Q_2)xy\, R(x, y)$$

iff

$$M \vDash Q_1 x\, Q_2 y \left(=\!(y) \wedge R(x, y)\right)$$

?

Generalized quantifiers in Dependence Logic

Lifting functions

The **Hodges space** of order ideals on the power set is

$$\mathcal{H}(A) = \mathcal{L}(\mathcal{P}(A)).$$

Given $h : \mathcal{P}(A) \to \mathcal{P}(B)$ we define the **Hodges lift**:

$$\mathcal{L}(h) : \mathcal{H}(A) \to \mathcal{H}(B), \mathscr{X} \mapsto \downarrow\{\, h(X) \mid X \in \mathscr{X} \,\},$$

where $\downarrow\mathscr{X}$ is the downward closure of $\mathscr{X}$, i.e.

$$\downarrow\mathscr{X} = \{\, X \mid \exists Y \in \mathscr{X}, X \subseteq Y \,\}.$$

Lifting quantifers

- Q a monotone type $\langle 1 \rangle$ quantifier.
- $Q : \mathcal{P}(M^{n+1}) \to \mathcal{P}(M^n)$
- $\mathcal{H}(Q) : \mathcal{H}(M^{n+1}) \to \mathcal{H}(M^n)$
- Gives truth conditions for Q in Hodges semantics:

  $$M \models_X Qx\,\varphi \text{ iff there is } F : X \to Q \text{ such that } M \models_{X[F/x]} \varphi.$$

  where $X[F/x] = \{\ s[a/x] \mid a \in F(s)\ \}$.
- $\mathcal{H}(\exists)$ and $\mathcal{H}(\forall)$ give the same truth conditions for $\exists$ and $\forall$ as before.

Proposition

For formulas φ without dependence atoms:

$$M \models_X \varphi \textit{ iff for all } s \in X, M \models \varphi[s].$$

Quantifiers and dependence

If Q contains no singletons then $M \not\models_X Qx\,(=(x) \wedge \varphi)$.
Assume that $D(x,y)$ is an atom closed under subteams satisfying:

$$\forall x\, Qy\,(D(x,y) \wedge R(x,y)) \leftrightarrow \mathrm{Br}(\forall, Q)xy\, R(x,y).$$

Fix $M = \{0,1,2\}$, then $(M, M^2) \models \mathrm{Br}(\forall, \exists^{\geq 3})xy\, R(x,y)$, thus: $M \models_{[M^2/x,y]} D(x,y)$. Using that D is closed under taking subteams:

$$X = (\{0,1\} \times \{0,1\}) \cup (\{2\} \times \{1,2\})$$

satisfies the atom D and thus $(M, X) \models \forall x\, \exists^{\geq 2} y\,(D(x,y) \wedge R(x,y))$.
However $(M, X) \not\models \mathrm{Br}(\forall, \exists^{\geq 3})xy\, R(x,y)$.
Thus no single atom $D(x,y)$ closed under taking subteams does the job intended with both the quantifiers $\exists^{\geq 2}$ and $\exists^{\geq 3}$.

Multivalued Dependencies

A course database

Course	Student	Credits
LC1510	Svensson	7.5 hp
LC1510	Johansson	7.5 hp
LC1520	Svensson	15 hp
LC1520	Andersson	15 hp

- $=(\text{Course}, \text{Credits})$
- It is not true that $=(\text{Course}, \text{Student})$.
- $=()$ is **context independent**: $X \models =(\bar{x})$ iff $Y \models =(\bar{x})$, where Y is X with some columns / variables, not in $\bar{x}$, removed.
- $=()$ is **closed downwards**: If $X \models =(\bar{x})$ then $Y \models =(\bar{x})$, where Y is X with some rows / assignments removed.

A course database

Course	Student	Credits	Year
LC1510	Svensson	7.5 hp	2010
LC1510	Johansson	7.5 hp	2011
LC1520	Svensson	15 hp	2011
LC1520	AnderssonJohansson	15 hp	2011

- F^{Student} takes values for Course and Credits and gives set of possible values for Student.
- $F^{\text{Student}}(\text{LC1510}, 7.5 \text{ hp}) = \{ \text{Svensson}, \text{Johansson} \}$
- F^{Student} is determined by the value of Course.
- $[\text{Course} \twoheadrightarrow \text{Student}]$
- $[\twoheadrightarrow]$ **dependent** on context.
- $F^{\text{Student}}(\text{LC1510}, 7.5 \text{ hp}, 2010) = \{ \text{Svensson} \}$
- $F^{\text{Student}}(\text{LC1510}, 7.5 \text{ hp}, 2011) = \{ \text{Johansson} \}$
- $[\twoheadrightarrow]$ **not** closed downwards.
- **not** $[\twoheadrightarrow \text{Student}]$

Multivalued dependence and teams

- If $s \in X$ then $F^y_X(s) = \{ a \mid s[a/y] \in X \}$.

Definition

$X \vDash [\bar{x} \twoheadrightarrow y]$ if F^y_X is determined by the values of $\bar{x}$. (Only for $y \notin \bar{x}$.)

Proposition

$X \vDash [\bar{x} \twoheadrightarrow y]$ iff for all $s, s' \in X$ such that $s(\bar{x}) = s'(\bar{x})$ there exists $s_0 \in X$ such that $s_0(\bar{x}) = s(\bar{x})$, $s_0(y) = s(y)$, and $s_0(\bar{z}) = s'(\bar{z})$, where $\bar{z}$ are the variables in $dom(X) \setminus (\{ \bar{x} \} \cup \{ y \})$.

- $X \vDash [\bar{x} \twoheadrightarrow y]$ is **dependent on context** and **not closed downwards**.
- $X \vDash =(\bar{x}, y)$ iff $X \vDash [\bar{x} \twoheadrightarrow y]$ and F^y_X only takes singleton values.

Generalized quantifiers and multivalued dependence

Proposition

If Q is monotone then $M \vDash Br(Q,Q)xy\, R(x,y)$ iff

$$M \vDash Qx\, Qy\, \big([\twoheadrightarrow y] \wedge R(x,y)\big).$$

- $\vDash \forall x\, [\twoheadrightarrow x]$, but $M \not\vDash \forall x {=}(x)$ for $|M| \geq 2$, thus $M \not\vDash \forall x \forall y\, ({=}(y) \wedge R(x,y))$.
- $\mathrm{Br}(\forall,\forall)xy\, R(x,y)$ is equivalent to $\forall x \forall y\, R(x,y)$.

Proposition

FOL + multivalued dependencies has the same strength, on the level of sentences, as ESO, and thus as Dependence Logic.

In a DL-formula with all occurrences of dependence atoms of the form $\exists y\big({=}(\bar{x},y) \wedge \cdot\big)$ we can replace the $=()$s with $[\twoheadrightarrow]$s keeping the meaning.

Axiomatization of multivalued dependence

- Fix a set V of variables. X is sets of assignments $s : V \to M$.
- $D \cup \{ d \}$ is a (finite) set of atoms of the form $[\bar{x} \twoheadrightarrow \bar{y}]$.
- $D \models d$ if for all X, $X \models D$ implies $X \models d$.

Proposition (Beeri, Fagin, Howard, 1977)

$D \models d$ iff d is derivable from D using the following inference rules:

- ***Complementation:*** *If $\bar{x} \cup \bar{y} \cup \bar{z} = V$, $\bar{y} \cap \bar{z} \subseteq \bar{x}$, and $[\bar{x} \twoheadrightarrow \bar{y}]$ then $[\bar{x} \twoheadrightarrow \bar{z}]$*
- ***Reflexivity:*** *If $\bar{y} \subseteq \bar{x}$ then $[\bar{x} \twoheadrightarrow \bar{y}]$.*
- ***Augmentation:*** *If $\bar{z} \subseteq \bar{w}$ and $[\bar{x} \twoheadrightarrow \bar{y}]$ then $[\bar{x}, \bar{w} \twoheadrightarrow \bar{y}, \bar{z}]$.*
- ***Transitivity:*** *If $[\bar{x} \twoheadrightarrow \bar{y}]$ and $[\bar{y} \twoheadrightarrow \bar{z}]$ then $[\bar{x} \twoheadrightarrow \bar{z} \setminus \bar{y}]$.*

Embedded multivalued dependence

- Multivalued dependence is dependent on context.

Definition

$X \models [\bar{x} \twoheadrightarrow \bar{y} | \bar{z}]$ iff $Y \models [\bar{x} \twoheadrightarrow \bar{y}]$ where Y is the projection of X onto $\{ \bar{x}, \bar{y}, \bar{z} \}$.

- $[\bar{x} \twoheadrightarrow \bar{y} | \bar{z}]$ is **independent on context**.
- Observe that this is the **independence atom**:

$$\bar{y} \perp_{\bar{x}} \bar{z} \text{ iff } [\bar{x} \twoheadrightarrow \bar{y} | \bar{z}]$$

- However, this relation is **not** axiomatizable. [Sagiv Walecka 1982]

To Do

- What are the definable sets of teams in multivalued dependence logic?
- What is the exact relationship with the Independence Logic of Grädel and Väänänen?
- Which generalized quantifiers Q have uniform definitions in Dependence Logic?
- What should $[t_1 \twoheadrightarrow t_2]$ mean?

Thank you for your attention.