Expansions omitting a type

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Prelimiaries

- All languages will be recursive, so will all extensions of languages.
- All models will be models of PA (even though many results work for arbitrary models).
- $M \models p\!\uparrow$ means that M omits the type $p(x)$.

Recursive saturation

- A *type* $p(x, a)$ over a model M is a set of formulas, with parameter $a \in M$, consistent with the theory of (M, a).
- M is *recursively saturated* if all recursive types over M are realized.
- Any model M has an elementary extension of the same cardinality which is recursively saturated.

Resplendence

- A model M is *resplendent* if for every recursive theory T, in bigger language, consistent with $\mathrm{Th}(M)$ there is an expansion of M satisfying T.
- Every countable recursively saturated model is resplendent, and every resplendent model is recursively saturated.

Arithmetic saturation

- A model M is arithmetically saturated if it is recursively saturated and $\mathrm{SSy}(M)$ is closed under the jump operator, i.e., $\mathrm{SSy}(M)$ is arithmetically closed.
- Kaye, Kossak and Kotlarski proved that a countable recursively saturated model of PA has a maximal automorphism iff the model is arithmetically saturated.
- A maximal automorphism is an automorphism moving all non-definable points.

Motivation

- A countable model of PA is recursively saturated iff it is resplendent.
- A countable model of PA is arithmetically saturated iff it has a maximal automorphism.
- Is there an expandability notion which is equivalent to arithmetic saturation?

Maximal automorphisms

- $f \in \mathrm{Aut}(M)$ is *maximal* if it moves all non-definable points, i.e. $\mathrm{fix}(f) = \mathrm{df}(0)$.
- There is a type $p(f, x)$ such that $f \in \mathrm{Aut}(M)$ is maximal iff M omits $p(f, x)$:

$$p(f, x) = \{\, f(x) = x \wedge t(0) \neq x \mid t \text{ is a Skolem term} \,\}$$

Transplendence

- Is there a notion of expandability that concludes that the expansion omits a type (and satisfies a theory)?
- Proposed notion is called transplendence. It needs a strong consistency notion to work.

Definitions and existence

- $\mathrm{Con}_\kappa(T + p{\uparrow}/T_0)$ holds if there is a κ-saturated model of T_0 with an expansion satisfying $T + p{\uparrow}$.
- $\mathrm{Con}(T + p{\uparrow}/M)$ holds if $\mathrm{Con}_{|M|}(T + p{\uparrow}/\,\mathrm{Th}(M))$.
- A model is *transplendent* if for every $T, p(x) \in \mathrm{SSy}(M)$ such that $\mathrm{Con}(T + p{\uparrow}/M)$ there exists an expansion of M satisfying $T + p{\uparrow}$.
- Any saturated model is transplendent.
- There are countable transplendent models.

Nonexistence

- If M is transplendent then $\mathrm{SSy}(M)$ is an elementary submodel of $\mathscr{P}(\omega)$ (as ω-standard models of second order arithmetic), i.e., $\mathrm{SSy}(M)$ is a β_ω model.
- That implies M being arithmetically saturated, but also much more saturation.
- For any $A \subseteq \omega$ let $\mathrm{tp}(A)$ be the type of A in $\mathscr{P}(\omega)$.
- If M is transplendent then for any $A \in \mathrm{SSy}(M)$ the type $\mathrm{tp}(A) \in \mathrm{SSy}(M)$.
- From certain set theoretic assumptions ($V = L$ or PD) this property implies the above property.

Subtransplendence

- A model M is *subtransplendent* if for every $T, p(x) \in \mathrm{SSy}(M)$ such that there exists a model of $\mathrm{Th}(M) + T + p\uparrow$ there exists an elementary submodel N of M and an expansion of N satisfying $T + p\uparrow$.
- A model is subtransplendent iff it is β saturated.
- $\mathrm{SSy}(M)$ is a β model if $\mathrm{SSy}(M)$ is elementary embedded for Σ^1_1 formulas in $\mathscr{P}(\omega)$.
- If $\mathrm{SSy}(M)$ is a β model and M is recursively saturated then we say that M is β saturated.
- β saturation is stronger than aritmetic saturation.
- Any transpledendent model is subtransplendent.

Limit transplendence

- A type $p(x)$ is a limit in T if there is no formula $\varphi(x)$ such that $T \models \forall x(\varphi(x) \to p(x))$ and $T + \exists x \varphi(x)$ is consistent.
- Limit types are much easier to handle than arbitrary types.
- A model M is *limit transplendent* if for every $T, p(x) \in \mathrm{SSy}(M)$ such that $p(x)$ is a limit in T and there exists a model of $\mathrm{Th}(M) + T + p\uparrow$ there exists an expansion of M satisfying $T + p\uparrow$.
- A countable arithmetically saturated model is limit transplendent.
- Is limit transcendence enough for arithmetic saturation?
- Yes, *if* we can prove that the type $p(f, x)$ saying that f is maximal is a limit type (in some recursive theory).

Summary

- Q: Is there an expandability notion which is equivalent to arithmetic saturation?
- Transplendence is too strong ($\Rightarrow$ SSy(M) is a β_ω model).
- Subtransplendence is too strong ($\Rightarrow$ SSy(M) is a β model).
- Limit tranplendence might be to weak ($\Leftarrow$ SSy(M) is arithmetically closed).

The end

That's it!