Variations on resplendency and recursive saturation

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These slides are available at:

`http://engstrom.morot.org/`

Preliminaries

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- All models will be models of PA^*, i.e., PA together with induction axioms for the full language.

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- Any model M has an elementary extension of the same cardinality which is recursively saturated.

...continued

- $\mathrm{SSy}(M) \subseteq \mathscr{P}(\omega)$ is the standard system of M, i.e., the collection of standard parts of parameter definable sets; i.e., the collection of all sets of the form $\{\, n \in \omega \mid M \models \varphi(n, a) \,\}$, where $a \in M$.

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- For any M, all recursive sets are in $\mathrm{SSy}(M)$.

A digression

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- Arithmetic saturation: rec sat plus $\mathrm{SSy}(M)$ closed under the jump operator.
- β-saturation: rec sat plus $\mathrm{SSy}(M)$ is a β-model, i.e., for every Σ^1_1-formula $\Theta(X)$ and $A \in \mathrm{SSy}(M)$; if $\mathbb{N}_2 \models \Theta(A)$ then $\mathrm{SSy}(M) \models \Theta(A)$.

$\mathbb{N}_2$ is the standard model of second-order arithmetic.

Arithmetic saturation

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First introduced by Kaye, Kossak and Kotlarski when they proved that a countable recursively saturated model of arithmetic has a maximal automorphism iff the model is arithmetically saturated.

A maximal automorphism is an automorphism moving all non-definable points.

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- However; Solovay later proved that no short cofinally expandable models exist.

...continued

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> For every $T, p(x,a) \in \mathrm{SSy}(M)$, where T is a theory and $p(x)$ is a type, both in a recursive extension of the language of (M,a), $a \in M$:
> If $\mathrm{Th}(M,a) + T + p{\uparrow}$ has a model then there is an elementary submodel $a \in L$ of M with an expansion satisfying $T + p{\uparrow}$.

$p{\uparrow}$ means that $p(x,a)$ is omitted.

Back from the digression

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- No model (M, st) is recursively saturated since the type

$$\{\, x > n \wedge \mathrm{st}(x) \mid n \in \omega \,\}$$

is omitted.

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- Any type over M (in which st does not occur) is a standard type.

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> for all standard types $p(x, a) \in \mathrm{SSy}(M)$ over (M, st) there is a complete standard type $q(x, a) \in \mathrm{SSy}(M)$ extending $p(x, a)$.

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$\Rightarrow$ Let M be std rec sat, and $p(x, a) \in \mathrm{SSy}(M)$ a std type. Let $m \in M$ realize $p(x, a)$. Then, $p(x, a) \subseteq tp_{(M,\mathrm{st})}(m/a) \in \mathrm{SSy}(M)$.

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$\Leftarrow$ By a Henkin type construction.

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(1) $\mathrm{SSy}(M)$ is a β_ω-model of second-order arithmetic, i.e., as second order models $\mathrm{SSy}(M) \prec \mathbb{N}_2$, where $\mathbb{N}_2$ is the standard second-order model of arithmetic.

(2) $\mathrm{SSy}(M)$ is closed under the following operation:

$$A \subseteq \omega \mapsto \mathrm{Th}(\mathbb{N}_2, A).$$

...continued

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- Under certain set-theoretic assumptions ($V = L$ or projective determinacy) we have $(2) \Rightarrow (1)$.
- Question: Are conditions (1) and (2) also sufficient, i.e., is any countable recursively saturated model satisfying condition (1) and (2) std rec sat?

The end

That’s all folks!