Logical constants: Invariance and definability

Fredrik Engström
Institut Mittag-Leffler

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- Everyone living in Djursholm is wealthy. I live in Djursholm. Therefore I'm wealthy.
- Everyone living in **Botkyrka** is wealthy. I live in **Botkyrka**. Therefore I'm wealthy.
- Everyone living in Djursholm is wealthy. **Björn** lives in Djursholm. Therefore **Björn** is wealthy.
- **Someone** living in Djursholm is wealthy. I live in Djursholm. Therefore I'm wealthy.

An "inferential" approach

$$\frac{\begin{array}{c}\forall x(Px \to Rx)\\ Pc\end{array}}{Rc}$$

$$\frac{\begin{array}{c}\forall x(Px \to Qx)\\ Pc\end{array}}{Qc}$$

$$\frac{\begin{array}{c}\forall x(Px \to Rx)\\ Pd\end{array}}{Rd}$$

$$\frac{\begin{array}{c}\forall x(Px \vee Rx)\\ Pc\end{array}}{Rc}$$

$$\neg \ldots (P \ldots R \ldots)$$

A "model theoretic" approach

An operator (function/predicate) is a logical constant if it is **topic neutral**.

- Examples: $\exists$, $\forall$, $\neg$, and $\rightarrow$.
- Non-example: "for all even numbers"
- Debatable: "for infinitely many", $=$

Mautner, Tarski, Mostowski, Lindenbaum: Logic is the the study of the invariants under the most general transformations (=permutations). (Klein's Erlangen program)

Definition (Lindström/Mostowski)

A (global) **generalized quantifier** *Q of type $\langle n_1, \ldots, n_k \rangle$ is a (class) of structures in the language $\{ R_1, \ldots, R_k \}$ where R_i is of arity n_i.*

Examples:

- $\exists = \{ (M, A) \mid A \subseteq M, A \neq \emptyset \}$
- $\forall = \{ (M, M) \mid M \}$
- $Q_0 = \{ (M, A) \mid A \subseteq M, |A| \geq \aleph_0 \}$
- $\exists^{=\kappa} = \{ (M, A) \mid A \subseteq M, |A| = \kappa \}$
- $I = \{ (M, A, B) \mid A, B \subseteq M, |A| = |B| \}$
- $W = \left\{ (M, R) \,\middle|\, R \subseteq M^2, R \text{ is well-founded} \right\}$
- $Q^A = \{ (M, B) \mid A \subseteq B \}$

- $\varphi(M) = \{ \bar{a} \in M^k \mid M \models \varphi(\bar{a}) \}$
- $M \models Qx_0 \ldots x_{k-1}\varphi(x_0, \ldots, x_{k-1})$ *iff* $(M, \varphi(M)) \in Q$ *(*Q *of type* $\langle k \rangle$*)*

Local versions: For a given domain M, let (for Q of type $(\langle k \rangle)$

$$Q_M = \left\{ R \subseteq M^k \;\middle|\; (M, A) \in Q \right\}.$$

A (local) quantifier Q_M of type $\langle k \rangle$ is definable in the logic $\mathscr{L}$ if there is φ of $\mathscr{L}$, such that

$$(M, R) \models \varphi \text{ iff } R \in Q_M.$$

Tarski's thesis

A (local) quantifier on a domain M is a logical constant iff it is invariant under all **permutations** *of M.*

Examples: $\exists, \forall, Q_0, \exists^{=\kappa}, I$
Non-examples: Q^A

Mostowski's thesis

A quantifier Q is a logical constant iff it is invariant under all **bijections** *(across domains).*

Theorem (McGee -91 / Krasner -38)

Q is bijection invariant iff for each κ there is a formula in $\mathscr{L}_{\infty\infty}$ defining Q_κ.

Fix a domain Ω. Quantifier means local quantifier on Ω.
$\mathscr{Q}$ is a set of quantifiers.
G subgroup of the full symmetric group S_Ω.

Definition

- *Let* $\mathrm{Aut}(\mathscr{Q})$ *be the group of all permutations of* Ω *fixing all quantifiers in* $\mathscr{Q}$*:*
  $\mathrm{Aut}(\mathscr{Q}) = \{\, g \in S_\Omega \mid g(Q) = Q \text{ for all } Q \in \mathscr{Q} \,\}$.
- *Let* $\mathrm{Inv}(G)$ *be the set of quantifiers fixed by* G*:*
  $\mathrm{Inv}(G) = \{\, Q \mid g(Q) = Q \text{ for all } g \in G \,\}$.

Theorem (Krasner/Bonnay/E)

- $\mathrm{Aut}(\mathrm{Inv}(G)) = G$
- $\mathrm{Inv}(\mathrm{Aut}(\mathscr{Q}))$ *is the set of quantifiers definable in* $\mathscr{L}_{\infty\infty}(\mathscr{Q})$

Proof

Aut(Inv(G)) $= G$: Let $\leq$ well-order Ω, and $Q = \{\, g(\leq) \mid g \in G \,\}$ of type $\langle 2 \rangle$. If $h \in \mathrm{Aut}(\mathrm{Inv}(G))$ then $h(\leq) \in Q$ and so there is $g \in G$ such that $h(\leq) = g(\leq)$, implying $h = g$.

Inv(Aut($\mathscr{Q}$)) **is the set of Qs definable in $\mathscr{L}_{\infty\infty}(\mathscr{Q})$**: We assume all quantifiers of type $\langle 1 \rangle$ and $\Omega = \omega$.
$Q' \in \mathrm{Inv}(\mathrm{Aut}(\mathscr{Q}))$ is defined by

$$\forall x_0, x_1, \ldots \Big[\bigwedge_{i \neq j} x_i \neq x_j \wedge \forall y \bigvee_i y = x_i \wedge \bigwedge_{Q \in \mathscr{Q}} \Big(\big(\bigwedge_{A \in Q} Qy \bigvee_{i \in A} y = x_i \big) \wedge \big(\bigwedge_{A \notin Q} \neg Qy \bigvee_{i \in A} y = x_i \big) \Big) \rightarrow \bigvee_{A \in Q'} \Big(\bigwedge_{i \in A} Px_i \wedge \bigwedge_{i \notin A} \neg Px_i \Big) \Big]$$

Theorem

If $\mathrm{Inv}_m(G)$ *are all* **monadic** *quantifiers invariant under* G *then there is a subgroup* G *such that* $\mathrm{Aut}(\mathrm{Inv}_m(G)) \supsetneq G$.

Proof. Let G be the group of **piecewise monotone** permutations on ω: $g \in S_\omega$ is piecewise monotone if there exists partitions $A_1 \cup \ldots \cup A_k = B_1 \cup \ldots \cup B_k = \omega$ such that $g|A_i$ is the unique increasing function $A_i \to B_i$.
$\mathrm{Aut}(\mathrm{Inv}_m(G))$ is closed in the topology generated by

$$U_{\bar{A},\bar{B}} = \{\, h \in S_\omega \mid h(A_i) = B_i \text{ all } i < k \,\}$$

as basic open sets, where $\bar{A} = A_0, \ldots, A_{k-1}$ and $\bar{B} = B_0, \ldots, B_{k-1}$ are subsets of ω.
The closure of G is S_ω.

Feferman's thesis -99

Definition

A (global) quantifier Q is invariant under **preimages of surjections** *if for every $h : M \to N$ surjection and for all $R \subseteq N^k$: $h^{-1}(R) \in Q_M$ iff $R \in Q_N$.*

Theorem (Feferman)

Quantifiers of type $\langle 1, \ldots, 1 \rangle$ are invariant under **preimages of surjections** *iff they are definable in $\mathscr{L}^{-}_{\omega\omega}$.*

Feferman's thesis

A quantifier is a logical constant iff it can de defined (in typed λ-calculus) from equality and monadic quantifiers invariant under preimages of surjections.

$h : M \to N$ can be "lifted" by: $h(Q_M) = \{ h(R) \mid R \in Q_M \}$.
Invariance under **surjections**: $h(Q_M) = Q_N$ for all surjective h.

Theorem (Casanovas -07)

"Quantifiers" are invariant under surjections iff they are definable in a certain positive fragment of $\mathscr{L}_{\omega\omega}$ (with restricted use of equality).

Invariance under **back-and-forth equivalence**: If (M, A) and (N, B) are back-and-forth equivalent, then $A \in Q_M$ iff $B \in Q_N$.

Theorem (Barwise -73)

A local quantifier Q on M is **back-and-forth invariant** *iff Q is definable in $\mathscr{L}_{\infty\omega}$.*

Bonnay (BSL -08) argues well for that **if** the logical constants are **the invariants** under some relation between structures, then this relation is **back-and-forth equivalence**.

Assume now all quantifiers are local quantifiers on ω.

Theorem (Lopez-Escobar)

A quantifier is **Borel** *and* **permutation invariant** *iff it is definable in* $\mathscr{L}_{\omega_1\omega}$*.*

Indicates a strong connection between $\mathscr{L}_{\omega_1\omega}$ and Borel quantifiers.

FALSE

Q is Borel and $\mathrm{Aut}(\mathscr{Q})$ *invariant iff Q is definable in* $\mathscr{L}_{\omega_1\omega}(\mathscr{Q})$*.*

Let $A \subseteq \omega$ be infinite and coinfinite and $Q' = \{ A \}$ then Q^A is $\mathrm{Aut}(Q')$ invariant, but not definable in $\mathscr{L}_{\omega_1\omega}(Q')$.

Theorem (E/Schlicht)

Let $\mathscr{Q}$ be a countable set of clopen quantifiers. Then Q is Borel and $\mathrm{Aut}(\mathscr{Q})$ invariant iff it is definable in $\mathscr{L}_{\omega_1\omega}(\mathscr{Q})$.

Question

For which sets $\mathscr{Q}$ of quantifiers does the theorem hold?

Thanks