A notion of recursive saturation for models of arithmetic with the standard predicate

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These slides are available at:

`http://engstrom.morot.org/`

Preliminaries

- All languages will be recursive extensions of the language of arithmetic:

$$\mathscr{L}_A = \{ +, \cdot, 0, 1, < \} .$$

- All models will be models of PA^*, i.e., PA together with induction axioms for the whole language.

Recursive saturation...

- A *type* $p(x, a)$ over a model M is a set of formulas with parameter $a \in M$, such that there is an elementary extension N of M and element $n \in N$ satisfying $N \models p(n, a)$.
- M is *recursively saturated* if all recursive types over M are realized.
- Any model M has an elementary extension of the same cardinality which is recursively saturated.

...continued

- $\mathrm{SSy}(M)$ is the standard system of M, i.e., the collection of standard parts of parameter definable sets; i.e., the collection of all sets of the form $\{\, n \in \omega \mid M \vDash \varphi(n, a) \,\}$, where $a \in M$.
- If M is recursively saturated then any type $p(x, a) \in \mathrm{SSy}(M)$ over M is realized in M.

The standard predicate

- The standard predicate, st, is the predicate of standard numbers.
- No model (M, st) is recursively saturated since the type

$$\{\, x > n \wedge \mathrm{st}(x) \,|\, n \in \omega \,\}$$

is omitted.

Standard recursive saturation

- A *standard type over* M is a type over (M, st) such that there is an ω-saturated elementary extension of M realizing the type.
- A model is *standard recursively saturated* (std rec sat) if all recursive standard types are realized.
- Any type over M (in which st is not mentioned) is a standard type.

An equivalence

A countable recursively saturated model is std rec sat iff

> for all standard types $p(x, a) \in \mathrm{SSy}(M)$ over (M, st) there is a complete standard type $q(x, a) \in \mathrm{SSy}(M)$ extending $p(x, a)$.

The proof

- Lemma: If M is countable and std rec sat, and $M \prec N$ is ω-saturated then $(M, \mathrm{st}) \prec (N, \mathrm{st})$.
- Thus; any type $\mathrm{tp}_{(M,\mathrm{st})}(m/a)$, where M is std rec sat, is a standard type.

$\Rightarrow$ Let M be std rec sat, and $p(x, a) \in \mathrm{SSy}(M)$ a std type. Let $m \in M$ realize $p(x, a)$. Then, $p(x, a) \subseteq tp_{(M,\mathrm{st})}(m/a) \in \mathrm{SSy}(M)$.

$\Leftarrow$ By a Henkin type construction.

The standard system...

Let M be a std rec sat model. Then

(1) $\mathrm{SSy}(M)$ is a β_ω-model of second-order arithmetic, i.e., as second order models $\mathrm{SSy}(M) \prec \mathbb{N}_2$, where $\mathbb{N}_2$ is the standard second-order model of arithmetic.

(2) $\mathrm{SSy}(M)$ is closed under the following operation:

$$A \subseteq \omega \mapsto \mathrm{Th}(\mathbb{N}_2, A).$$

...continued

- Under certain set-theoretic assumptions ($V = L$ or projective detereminacy) we have $(2) \Rightarrow (1)$.
- Question: Are conditions (1) and (2) also sufficient, i.e., is any countable recursively saturated model satisfying condition (1) and (2) std rec sat?

The end

That's all folks!