Classification problems and models of arithmetic

Fredrik Engström

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Introduction

- How hard is it to decide $M \cong N$ for $M, N \models T$?
- Finite M and N $\Rightarrow$ coded as natural numbers $\Rightarrow$ $\cong$ relation on ω.
- Models with domain ω in a relational language L is a point in the logic space X_L:

Definition

For a relational language L the logic space is

$$X_L = \prod_{R \in L} 2^{\omega^{a(R)}}.$$

- $\cong$ is an equivalence relation on X_L.

Reductions

Definition

Given equivalence relations E, F on X, Y respectievly we define $E \leq Y$ iff there is a function $f : X \to Y$ such that

$$x \, E \, x' \text{ iff } f(x) \, F \, f(x').$$

Not very interesting: $E \leq F$ iff $|X/E| \leq |Y/F|$.
Need to put restrictions on the function f.

Topology

Definition

Topology: (X, τ) *where* τ *is the set of open subsets of* X*:*

- $\emptyset, X \in \tau$
- $U, V \in \tau$ *then* $U \cap V \in \tau$
- $A \subseteq \tau$ *then* $\bigcup A \in \tau$.

Definition

X *is* **Polish** *if separable and completely metrizable.*

Examples:

- Countable spaces with the discrete topology.
- The Baire space $\mathscr{N} = \omega^\omega$. $\{ f \mid f(n) = k \}$
- The Cantor space $\mathscr{C} = 2^\omega$. $\{ f \mid f(n) = k \}$
- The logic space X_L. $\left\{ M \mid (\neg)R^M(\bar{a}) \right\}$
- $\mathbb{R}^n$

Borel sets

Definition

*Let X be a Polish space. A **σ-algebra** over X is a boolean algebra over X closed under countable unions. The least σ-algebra over X containing the open sets is the collection of **Borel sets**.*

Definition

$f : X \to Y$ *is* **continuous** *if $f^{-1}(U)$ is open for every open U. f is* **Borel** *if $f^{-1}(U)$ is Borel for every open U.*

Theorem (Luzin-Suslin)

If f is Borel and 1-to-1 then $f(A)$ is Borel for every Borel A.

The standard Borel space

Definition

$X \cong_B Y$ if there is a Borel bijection from X to Y.

Theorem

If X and Y are Polish uncountable spaces then $X \cong_B Y$.

Thus, all uncountable Polish spaces give rise to the same Borel space: **the standard Borel space**. (Observe that any uncountable Polish space is of size $2^{\aleph_0}$.)

Borel and language

Theorem

A set $A \subseteq X_L$ is Borel iff there is an $L_{\omega_1\omega}$-formula φ with natural number parameters in the signature L such that

$$M \models \varphi \text{ iff } M \in A.$$

Theorem

A set $X \subseteq \mathcal{N}$ is Borel iff there is a Σ^1_1-formula σ and a Π^1_1-formula π with function parameters in the signature of arithmetic with one function symbol added such that

$$\mathbb{N} \models \forall f(\sigma \leftrightarrow \pi) \text{ and } (\mathbb{N}, f) \models \sigma \text{ iff } f \in X.$$

Fact: $\mathrm{WO} \subseteq 2^{\omega^2}$ is not Borel.

Borel reductions

Definition

Let X, Y be Polish spaces and E, F equivalence relations on X, Y respectively.

- $E \leq_B F$ *iff* $E \leq F$ *by a Borel function* $f : X \to Y$.

Examples of equivalence relations:

- $\mathrm{id}(\omega)$
- $\mathrm{id}(2^\omega)$
- $f \; E_0 \; g$ if $f = g$ almost everywhere.
- $f =^+ g$ if $\mathrm{rg}(f) = \mathrm{rg}(g)$.
- $M \cong N$ over X_L.

$$\mathrm{id}(\omega) <_B \mathrm{id}(2^\omega) <_B E_0(\mathscr{C}) \equiv_B E_0(\mathscr{N}) <_B =^+ <_B \cong$$

Smooth relations

Definition

E is **completely classifiable** *or* **smooth** *if $E \leq_B id(2^\omega)$.*

Theorem (Silver)

If E is smooth then either $E \equiv_B id(2^\omega)$ or $E \leq_B id(\omega)$.

Theorem (Harrington-Kechris-Louveau)

If E is Borel and non-smooth then $E_0 \leq_B E$.

Countable relations

Definition

E is said to be **essentially countable** *if there is E' with only countable equivalence classes such that $E \equiv_B E'$.*

Theorem

There is a universal countable Borel relation E_∞, i.e., E is essentially countable iff $E \leq_B E_\infty$.

Theorem

$E_\infty <_B =^+$

Not known if there exists E such that $E_\infty <_B E <_B =^+$.

The isomorphism relation

Given Θ $L_{\omega_1\omega}$-sentence let $\cong_\Theta$ be the isomorphism relation on the class of countable models of Θ.

Definition

$Mod(\Theta)$ is **Borel complete** *if for any Θ' in any L' $\cong_{\Theta'} \leq_B \cong_\Theta$.*

Examples of Borel complete classes:

- The class of linear orders.
- The class of groups.
- The class of connected graphs.

Observe that the isomorphism relation on a Borel complete class is **not** Borel.

Models of arithmetic

Let T be a completion of PA.

Theorem

$Mod(T)$ is Borel complete.

Theorem (Gaifman)

For linear orders I we can build a (Gaifman) model $M_T(I) \models T$ such that $I \cong J$ iff $M_T(I) \cong M_T(J)$.

Proof. Let $p(x)$ be a minimal type over T and let $M_T(I)$ be the Skolem closure of some sequence $c_i, i \in I$ satisfying

$$\bigcup_{i \in I} p(c_i) \cup \bigcup_{i<j \in I} c_i < c_j.$$

$p(x)$ can be constructed arithmetically in T. Thus, for countable linear orders $M_T(I)$ is arithemtical in T and I. Therefore Borel.

Finitely generated models of arithmetic

M, N finitely generated then $M \cong N$ iff there are generators a, b of M, N respectively such that $\mathrm{tp}(a) = \mathrm{tp}(b)$.

Theorem

Every countable model M has a finitely generated (superminimal) elementary end-extension.

Theorem (Schmerl)

The isomorphism relation $\cong^{\mathrm{fg}}$ of finitely generated models of T is

- *Borel,*
- *essentially countable, and*
- *non-smooth.*

Conjecture: $\cong^{\mathrm{fg}} \equiv_B E_\infty$

Recursively saturated models of arithmetic

Theorem

Countable recursively saturated models of arithmetic are precisely characterized by their standardsystem.

There is a Borel function $f : X_L \to (2^\omega)^\omega$ taking M to an ordered sequence of the standard system of M.

Theorem

$\cong_T^{rs} \equiv_B =^+$.

f above proves $\cong_T^{\mathrm{rs}} \leq_B =^+$.

Thanks