Generalized quantifiers in dependence logic

Fredrik Engström

2011–02–04

1. Dependencies
2. Logics with dependence
3. Generalized quantifiers in dependence friendly settings

Dependencies

Functional dependence

Course	Book	Lecturer
LC1510	Mendelson	Engström
LC1510	Mendelson	Kaså
LC1520	Halmos	Engström

$$X \models [\text{Course} \rightarrow \text{Book}]$$

$$X \not\models [\text{Course} \rightarrow \text{Lecturer}]$$

Closed downwards: If $X \subseteq Y$ and $Y \models [\bar{x} \rightarrow \bar{y}]$ then $X \models [\bar{x} \rightarrow \bar{y}]$.
Closed under projections: If Y is X with one column/variable/attribute z deleted, $Y \models [\bar{x} \rightarrow \bar{y}]$ and $z \notin \bar{x} \cup \bar{y}$ then $Y \models [\bar{x} \rightarrow \bar{y}]$.

Multivalued dependence

Course	Book	Lecturer
LC1510	Mendelson	Engström
LC1510	Mendelson	Kaså
LC1520	Halmos	Engström
LC1520	Mendelson	Engström
LC1520	Halmos	Kaså
LC1520	Mendelson	Kaså

$$F^{\text{Book}}(\text{LC1510}, \text{Engström}) = \{\,\text{Mendelson}\,\}$$
$$F^{\text{Book}}(\text{LC1510}, \text{Kaså}) = \{\,\text{Mendelson}\,\}$$
$$F^{\text{Book}}(\text{LC1520}, \text{Engström}) = \{\,\text{Mendelson}, \text{Halmos}\,\}$$
$$F^{\text{Book}}(\text{LC1520}, \text{Kaså}) = \{\,\text{Mendelson}, \text{Halmos}\,\}$$

Multivalued dependence

Course	Book	Lecturer
LC1510	Mendelson	Engström
LC1510	Mendelson	Kaså
LC1520	Halmos	Engström
LC1520	Mendelson	Engström
LC1520	Halmos	Kaså
LC1520	Mendelson	Kaså

F^{Book} only depends on the value of Course. Therefore,

$$X \models [\text{Course} \twoheadrightarrow \text{Book}] .$$

$X \models [\bar{x} \twoheadrightarrow y]$ iff F^y_X only depends on the values of $\bar{x}$.
$X \models [\bar{x} \twoheadrightarrow \bar{y}]$ is **not** closed downwards or under projections.

Teams and dependencies

- Fix a domain M and a set of variables U.
- A team X is a set of assignments of elements of M to U, i.e., $X \subseteq M^U$.
- $X \vDash [\bar{x} \rightarrow \bar{y}]$ if for all $s, s' \in X$ if $s(\bar{x}) = s'(\bar{x})$ then $s(\bar{y}) = s'(\bar{y})$.
- $X \vDash [\bar{x} \twoheadrightarrow \bar{y}]$ if for all $y \in \bar{y}$ F_X^y depends only on the values of $\bar{x}$.

Proposition

$X \vDash [\bar{x} \twoheadrightarrow \bar{y}]$ iff for all $s, s' \in X$ such that $s(\bar{x}) = s'(\bar{x})$ there exists $s_0 \in X$ such that $s_0(\bar{x}) = s(\bar{x})$, $s_0(\bar{y}) = s(\bar{y})$, and $s_0(\bar{z}) = s'(\bar{z})$, where $\bar{z}$ is $U \setminus (\bar{x} \cup \bar{y})$.

Axiomatization of functional dependence

$D \cup \{ \varphi \}$ is a (finite) set of functional dependence atoms.

$$D \vDash \varphi \text{ if } \forall X (X \vDash D \Rightarrow X \vDash \varphi).$$

Proposition (Armstrong 1974)

$D \vDash \varphi$ iff φ is derivable from D with the rules:

- **Reflexivity:** *If $\bar{y} \subseteq \bar{x}$ then $[\bar{x} \to \bar{y}]$.*
- **Augmentation:** *If $[\bar{x} \to \bar{y}]$ then $[\bar{x}, \bar{z} \to \bar{y}, \bar{z}]$.*
- **Transitivity:** *If $[\bar{x} \to \bar{y}]$ and $[\bar{y} \to \bar{z}]$ then $[\bar{x} \to \bar{z}]$.*

Axiomatization of multivalued dependence

Fix a set U of variables.

Proposition (Beeri, Fagin, Howard, 1977)

Then $D \models \varphi$ iff φ is derivable from D with the following inference rules:

- **Complementation:** *If $\bar{x} \cup \bar{y} \cup \bar{z} = U$, $\bar{y} \cap \bar{z} \subseteq \bar{x}$, and $[\bar{x} \twoheadrightarrow \bar{y}]$ then $[\bar{x} \twoheadrightarrow \bar{z}]$*
- **Reflexivity:** *If $\bar{y} \subseteq \bar{x}$ then $[\bar{x} \twoheadrightarrow \bar{y}]$.*
- **Augmentation:** *If $\bar{z} \subseteq \bar{w}$ and $[\bar{x} \twoheadrightarrow \bar{y}]$ then $[\bar{x}, \bar{w} \twoheadrightarrow \bar{y}, \bar{z}]$.*
- **Transitivity:** *If $[\bar{x} \twoheadrightarrow \bar{y}]$ and $[\bar{y} \twoheadrightarrow \bar{z}]$ then $[\bar{x} \twoheadrightarrow \bar{z} \setminus \bar{y}]$.*

Logics with dependence

Motivation

Some relative of each villager and some relative of each townsmen hate each other. (Hintikka 1974)

$$\begin{matrix} \forall x \exists y \\ \forall z \exists w \end{matrix} R(x,y,z,w)$$

$$\exists f, g \forall x, z \; R(x, f(x), z, g(z)).$$

Most of the dots and most of the stars are all connected by lines. (Barwise 1979)

Branching

For monotone quantifiers the branching of Q_1 and Q_2

$$\begin{matrix} Q_1 x \\ Q_2 y \end{matrix} R(x,y)$$

should be interpreted as

$$\mathrm{Br}(Q_1, Q_2)xy\, R(x,y),$$

where $\mathrm{Br}(Q_1, Q_2)$ is the quantifier

$$\{\, R \mid \exists A \in Q_1, B \in Q_2, A \times B \subseteq R \,\}.$$

Dependence logic

- Syntax of dependence logic: FOL + $[t_1, \ldots, t_{k-1} \rightarrow t_k]$.
- Assume all formulas in negation normal form.
- Let $X \subseteq M^{\{\bar{x}\}}$. We write $M \vDash_X \varphi$ where the free variables of φ are among $\bar{x}$.
- Each $s \in X$ is an assignment of values to the variables in $\bar{x}$.
- For FOL-formulas φ we have $M \vDash_X \varphi$ iff $M \vDash \varphi[s]$ for every $s \in X$.

Semantics

- $M \vDash_X [\bar{x} \to \bar{y}]$ if $X \vDash [\bar{x} \to \bar{y}]$
- $M \vDash_X \varphi \vee \psi$ if there are Y and Z such that $X = Y \cup Z$ and $M \vDash_Y \varphi$ and $M \vDash_Z \psi$.
- $M \vDash_X \exists x \varphi$ if there is a function $f : X \to M$ such that $M \vDash_{X[f/x]} \varphi$, where $X[f/x] = \{ s[f(s)/x] \mid s \in X \}$.
- $M \vDash_X \forall x \varphi$ if $M \vDash_{X[M/x]} \varphi$, where $X[M/x] = \{ s[a/x] \mid a \in M, s \in X \}$.

Some facts of DL

- **Empty team:** $M \vDash_{\emptyset} \varphi$ for any φ.
- **LEM:** There are sentences σ such that $M \nvDash \sigma \vee \neg\sigma$.
- **Monotonicity:** If $M \vDash_X \varphi$ and $Y \subseteq X$ then $M \vDash_Y \varphi$.
- **Weakness:** For sentences σ there is translation $\hat{\sigma}$ to Σ^1_1 such that $\sigma \equiv \hat{\sigma}$.
- **Strength:** For Σ^1_1 sentences Φ there is a translation $\hat{\Phi}$ to DL such that $\Phi \equiv \hat{\Phi}$.

Generalized quantifiers in dependence friendly settings

Lifting

$$\mathscr{H}(M^n) = \mathscr{L}(\mathscr{P}(M^n))$$

Given $h : \mathscr{P}(A) \to \mathscr{P}(B)$ we define the Hodges-lift of that function as:

$$\mathscr{L}(h) : \mathscr{H}(A) \to \mathscr{H}(B), \mathscr{X} \mapsto \downarrow \{ h(X) \,\big|\, X \in \mathscr{X} \},$$

where $\downarrow \mathscr{X}$ is the downward closure of $\mathscr{X}$, i.e.,

$$\downarrow \mathscr{X} = \{ X \mid \exists Y \in \mathscr{X}, X \subseteq Y \}.$$

Lifting quantifers

Q a monotone type $\langle 1 \rangle$ quantifier.
$Q : \mathscr{P}(M^{n+1}) \to \mathscr{P}(M^n)$
$\mathscr{H}(Q)$ gives us truth condition for Q in Hodges semantics:

$$M \vDash_X Qx\varphi \text{ iff there is } F : X \to Q \text{ such that } M \vDash_{X[F/x]} \varphi.$$

where $X[F/x] = \{\, s[a/x] \,\big|\, a \in F(s) \,\}$.
$\mathscr{H}(\exists)$ and $\mathscr{H}(\forall)$ give us the same truth condition for $\exists$ and $\forall$ as before.

Proposition

For $L(Q)$-formulas φ and teams X we have $M \vDash_X \varphi$ iff for all $s \in X$, $M \vDash_s \varphi$.

Quantifiers and dependence

If Q contains no singletons then

$$M \not\models_X Qx([\rightarrow x] \wedge \varphi)$$

Assume that $D(x,y)$ is an atom closed under subteams satisfying:

$$\forall x Q y (D(x,y) \wedge R(x,y)) \longleftrightarrow \mathrm{Br}(\forall, Q)xy\ R(x,y).$$

Fix $M = \{0,1,2\}$, then $(M, M^2) \models \mathrm{Br}(\forall, \exists^{\geq 3})xy\ R(x,y)$, thus: $M \models_{[M^2/x,y]} D(x,y)$. By the closureness of D:

$$X = (\{0,1\} \times \{0,1\}) \cup (\{2\} \times \{1,2\})$$

satisfies the atom D and thus $(M,X) \models \forall x \exists^{\geq 2} y (D(x,y) \wedge R(x,y))$. However $(M,X) \not\models \mathrm{Br}(\forall, \exists^{\geq 3})xy\ R(x,y)$. Thus no atom $D(x,y)$ closed under taking subteams works as intended on both the quantifiers $\exists^{\geq 2}$ and $\exists^{\geq 3}$.

Quantifiers and multivalued dependence

Proposition

If Q is monotone then $M \models Br(Q,Q)xy\,R(x,y)$ iff $M \models QxQy([\twoheadrightarrow y] \wedge R(x,y))$.

$\models \forall x\,[\twoheadrightarrow x]$, but $M \not\models \forall x\,[\rightarrow x]$ for $|M| \geq 2$.
Thus $M \not\models \forall x \forall y([\rightarrow y] \wedge R(x,y))$.
$\mathrm{Br}(\forall,\forall)xyR(x,y)$ is equivalent to $\forall x \forall y R(x,y)$ and can thus be true.
FOL$+[\twoheadrightarrow]=$MVDL.

Proposition

MVDL has the same strength (on the level of sentences) as ESO.

Open Qs

What is the strength of FOL+$[\twoheadrightarrow]$ on formula level? Solved by V+K for DL.

MVDL+Q, where Q is ESO-definable is of the same strength as ESO. However, is there a way of uniformly define Q in MVDL?What about $Q = Q_0$?

Lunch