Generalized quantifiers in dependence logic

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Outline of dependence logic

Dependence logic

- Syntax of dependence logic: FOL $+\ t_1, \ldots, t_{k-1} \rightsquigarrow t_k$.
- Assume all formulas in negation normal form.
- Let $X \subseteq M^k$. We write $M \models_{\bar{x}/X} \varphi$ where the free variables of φ are among $\bar{x}$ (and the length of $\bar{x}$ is k).
- For each $s \in X$ we get an assignment $\bar{x}/s$ by assigning s_k to x_k.
- For FOL-formulas φ we have $M \models_{\bar{x}/X} \varphi$ iff $M \models_{\bar{x}/s} \varphi$ for every $s \in X$.

Definition of $\models$

- $M \models_{\bar{x}/X} P(\bar{t})$ iff $\forall s \in X, M \models_{\bar{x}/s} P(\bar{t})$
- $M \models_{\bar{x}/X} \neg P(\bar{t})$ iff $\forall s \in X, M \models_{\bar{x}/s} \neg P(\bar{t})$
- $M \models_{\bar{x}/X} \bar{t} \rightsquigarrow t'$ iff $\forall s, s' \in X (\forall i (t_i^{M,\bar{x}/s} = t_i^{M,\bar{x}/s'}) \Rightarrow t'^{M,\bar{x}/s} = t'^{M,\bar{x}/s'})$
- $M \models_{\bar{x}/X} \neg\bar{t} \rightsquigarrow t'$ iff $X = \emptyset$
- $M \models_{\bar{x}/X} \varphi \wedge \psi$ iff $M \models_{\bar{x}/X} \varphi$ and $M \models_{\bar{x}/X} \psi$
- $M \models_{\bar{x}/X} \varphi \vee \psi$ iff $\exists Y \cup Z = X, M \models_{\bar{x}/Y} \varphi$ and $M \models_{\bar{x}/Z} \psi$
- $M \models_{\bar{x}/X} \exists y \varphi$ iff $\exists F{:}X \to M, M \models_{\bar{x},y/X[F]} \varphi$
- $X[F] = \{ s, F(s) \mid s \in X \}$
- $M \models_{\bar{x}/X} \forall y \varphi$ iff $M \models_{\bar{x},y/X[M]} \varphi$
- $X[M] = \{ s, a \mid s \in X, a \in M \}$
- $M \models_X \varphi$

Some facts of DL

- Empty team: $M \models_{\emptyset} \varphi$ for any φ.
- $\neg$ LEM: There are sentences σ such that $M \not\models \sigma \vee \neg\sigma$. (Negation is game-theoretic negation.)
- Monotonicity: If $M \models_X \varphi$ and $Y \subseteq X$ then $M \models_Y \varphi$.
- Weakness: For sentences σ there is translation $\hat{\sigma}$ to Σ^1_1 such that $\sigma \equiv \hat{\sigma}$.
- Strength: For Σ^1_1 sentences Φ there is a translation $\hat{\Phi}$ to DL such that $\Phi \equiv \hat{\Phi}$.
- Thus, there is a sentence σ such that $(M, R) \models \sigma$ iff R is not well-founded on M. (However, being well-founded is not axiomatizable.)
- There is a formula $\mathrm{Tr}(x)$ of $\mathscr{L}_A$ such that PA $\models \mathrm{Tr}(\sigma) \leftrightarrow \sigma$ for all sentences σ. (Remember that LEM is not valid.)

First-order generalized quantifiers

The semantic setup

- Fix a model M.
- $[\bar{x}|\varphi] = \{ \bar{a} \in M \mid M \models_{\bar{x}/\bar{a}} \varphi \} \in \mathscr{P}(M^k)$
- Observe that $[\epsilon|\sigma]$ is $T = \{ \epsilon \}$ or $F = \emptyset$.
- A $\langle k_1, \ldots, k_l \rangle$-quantifier (on M) is a subset of $\mathscr{P}(M^{k_1}) \times \ldots \times \mathscr{P}(M^{k_l})$.
- Given a $\langle k \rangle$-quantifier Q we define

$$\phi^Q : R \mapsto \{ \bar{a} \mid R_{\bar{a}} \in Q \}, \mathscr{P}(M^{n+k}) \to \mathscr{P}(M^n).$$

- $R_{\bar{a}} = \{ \bar{b} \mid \bar{a}, \bar{b} \in R \}$
- Example: $\wedge = \{ \langle \top, \top \rangle \}$, $\phi^{\wedge}(R, S) = R \cap S$.

Quantifers as predicates or operators

- Is there any significant difference in looking at quantifiers as Q or ϕ^Q?
- Casanovas has proved that the Feferman-idea of homomorphism-invariant quantifiers is dependent on if we regard quantifiers as predicates or operators/functions.
- This seems to be the only instance where it matters (?).

Generalized quantifiers in DL

Semantics revisited

- Let $[\bar{x}|\varphi]_M = \{\, X \mid M \models_{\bar{x}/X} \varphi \,\}$
- $[\bar{x}|\varphi \wedge \psi] = [\bar{x}|\varphi] \cap [\bar{x}|\psi]$
- $[\bar{x}|\varphi \vee \psi] = \{\, Y \cup Z \mid Y \in [\bar{x}|\varphi], Z \in [\bar{x}|\psi] \,\}$
- $[\bar{x}|\exists y \varphi] = \{\, X \mid \exists F : X \to M, X[F] \in [\bar{x}, y|\varphi] \,\}$
- $[\bar{x}|\forall y \varphi] = \{\, X \mid X[M] \in [\bar{x}, y|\varphi] \,\}$
- The set $[\varphi]$ is an order ideal in the po-set $(\mathscr{P}(M^k), \subseteq)$.
- $\emptyset \in [\varphi]$.

The semantic setup

- $T_k = \mathscr{P}(M^k)$
- $S_k = \{ \mathscr{X} \subseteq T_k \mid \mathscr{X} \text{ order ideal} \}$. ($\mathscr{X}$ order ideal if $Y \subseteq X \in \mathscr{X}$ implies $Y \in \mathscr{X}$.)
- $S_0 = \{ \emptyset = \bot, \{ \emptyset \} = 0, \{ \emptyset, \{ \epsilon \} \} = \top \}$.
- To get a two-valued logic we could demand all the ideals in S_k to include $\emptyset$. (That would not make $\multimap$ fit in the framework.)
- By easing on the definition of S_k we could make "real" negation part of the framework, however the analysis would be much more cumbersome.

Quantifers

- A $\langle k_1, \ldots, k_l \rangle$ (local) quantifier $\mathtt{Q}$ is a function $\mathtt{Q} : S_{k_1} \times \ldots \times S_{k_l} \rightarrow S_0$.
- A $\langle k_1, \ldots, k_l \rangle$ (local) operator Φ, gives for each n a function $\Phi_n : S_{n+k_1} \times \ldots \times S_{n+k_l} \rightarrow S_n$.
- We say that a quantifier $\mathtt{Q}$ (operator Φ) is two-valued if for every $\emptyset \in \mathscr{X}$ we have $\emptyset \in \mathtt{Q}\mathscr{X}$ ($\emptyset \in \Phi\mathscr{X}$).
- Let $X \in T_k$ and $F : X \rightarrow T_1$ any function, then we can define the extension of X by F through $X[F] = \{ s, a \mid s \in X, a \in F(s) \}$.
- If $F : X \rightarrow T_1$ is the constant function taking value M we have $X[F] = X[M]$.
- Observe that if $X = \emptyset$ then $X[F] = \emptyset$ for the only function $F : X \rightarrow T_1$.

Examples of quantifiers

- $\exists(\mathscr{X}) = \{ X \in T_k \mid \exists F : X \to \mathscr{P}_1(M), X[F] \in \mathscr{X} \}$
- $\mathscr{P}_1(M) = \{ \{ a \} \mid a \in M \}$.
- $\forall(\mathscr{X}) = \{ X \in T_k \mid \exists F : X \to \{M\}, X[F] \in \mathscr{X} \}$
- $\forall^1(\mathscr{X}) = \{ X \in T_k \mid \forall F : X \to 1 \to \mathscr{P}_1(M), X[F] \in \mathscr{X} \}$
- $F : A \to 1 \to B$ means that $F : A \to B$ factors through the one element set 1 (i.e., F is constant).
- $\exists^1(\mathscr{X}) = \{ X \in T_k \mid \exists F : X \to 1 \to \mathscr{P}_1(M), X[F] \in \mathscr{X} \}$
- $\exists_0 = \exists_0^1$
- $\forall_0 \neq \forall_0^1$

Translations: DL $\leftrightarrow$ FOL

- $\mathsf{I} : T_k \to S_k$, $\mathsf{I}(X) = \{ Y \mid Y \subseteq X \}$ and $\cup : S_k \to T_k$.
- $\cup \circ \mathsf{I} = \mathrm{id}$ but $\mathsf{I} \circ \cup \neq \mathrm{id}$.
- $I(T) = \top$, $\cup \top = T$, $I(F) = 0$, $\cup 0 = F$.
- Given an operator $\Phi : S_{k+1} \to S_k$ let the flattening $\downarrow\Phi$ of Φ be defined by $\downarrow\Phi(X) = \cup\Phi(\mathsf{I}(X))$.
- Thus $\downarrow\Phi$ is a first-order $\langle 1 \rangle$-operator.
- The lift of a first-order operator ϕ is the two-valued operator $\uparrow\phi(\mathscr{X}) = \mathsf{I}(\phi(\cup\mathscr{X}))$.
- Flattening and lifting works for arbtrary operators as well.
- Therefore $(\downarrow\uparrow\phi)(X) = \cup\big((\uparrow\phi)(\mathsf{I}(X))\big) = \cup\big(\mathsf{I}(\phi(\cup\,\mathsf{I}(X)))\big) = \phi(X)$, so $\downarrow\uparrow\phi = \phi$.
- But in general $\uparrow\downarrow\Phi \neq \Phi$.
- $\uparrow\downarrow$ is a closure operator on the space of operators.

Principal operators

Those Φ for which $\uparrow\downarrow\Phi = \Phi$ we call principal.

Proposition

If L consists of principal operators then $L \equiv L'$ where L' is the first-order logic in which every operator has been flattened.

Proof.

Let $\downarrow\varphi$ be the first-order formula we get from φ be flattening all logical constants in φ. By induction we prove that for any φ:

$$M \models_{\bar{x}/X} \varphi \text{ iff } M \models_{\bar{x}/s} \downarrow\varphi \text{ for every } s \in X.$$

$\square$

Some reverse? Check which of the four quantifiers are principal.

Translations: quantifiers $\leftrightarrow$ operators

First-order $\langle 1 \rangle$-quantifier Q, define operators $\Phi_k^Q : S_{k+1} \to S_k$ by $\Phi_k^Q(\mathscr{X}) = \{ X \in T_k \mid \exists F : X \to Q, X[F] \in \mathscr{X} \}$.

Proposition

$$\Phi^{\exists} = \exists \textit{ and } \Phi^{\forall} = \forall$$

Recall:

$$\phi^Q(R) = \{ \bar{a} \mid R_{\bar{a}} \in Q \}$$

or equivalently (for monotone quantifier Q):

$$\phi^Q(R) = \{ \bar{a} \mid \exists F : \{\bar{a}\} \to Q, \{\bar{a}\}[F] \subseteq R \}$$

Connectives

A connective is an operator of type $\langle 0, \ldots, 0 \rangle$.

- $\mathscr{X} \vee \mathscr{Y} = \mathscr{X} \cup \mathscr{Y}$
- $\mathscr{X} \wedge \mathscr{Y} = \mathscr{X} \cap \mathscr{Y}$
- $\mathscr{X} \otimes \mathscr{Y} = \{ X \cup Y \mid X \in \mathscr{X}, Y \in \mathscr{Y} \}$
- $\mathscr{X} \oplus \mathscr{Y} = \{ Z \mid \forall X \cup Y = Z, X \in \mathscr{X}, Y \in \mathscr{Y} \}$
- $\mathscr{X} \rightarrow \mathscr{Y} = \{ Z \mid \mathscr{P}(Z) \cap \mathscr{X} \subseteq \mathscr{Y} \}$
- $\mathscr{X} \multimap \mathscr{Y} = \{ Z \mid \forall X \in \mathscr{X}, Z \cup X \in \mathscr{Y} \}$
- $\neg \mathscr{X} = \downarrow (\cup \mathscr{X})^c$

Non-example: $\sim \mathscr{X} = \{ X \mid X \notin \mathscr{X} \}$
Not truth-functions.

Negation

In this setting there is no way of treating (game theoretic) negation as a quantifier/operator: Assume $|M| > 1$. Let σ be $\forall x(\epsilon \rightsquigarrow x)$ then

$$[\sigma] = \{ \emptyset \}$$

$$[\neg\sigma] = [\exists x \neg(\epsilon \rightsquigarrow x)] = \{ \emptyset \}$$

Let τ be $\exists x(x \neq x)$ then

$$[\tau] = \{ \emptyset \}$$

$$[\neg\tau] = [\forall x(x = x)] = \{ \emptyset, \{ \epsilon \} \}$$

To make negation compositional in this sense it seems necessary to complicate the semantic framework considerably.

Atoms

- k-ary atoms $\leftrightarrow$ first-order $\langle k \rangle$-quantifiers.
- $=(R)$ iff $R \subseteq =$
- $\rightsquigarrow (R)$ iff $R \subseteq \mathrm{graph}(f)$ for some function f.

Lunch