Dependence and logicality

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LINT: Logic for INTeraction

- DEPLOG (Amsterdam, Gothenburg, Helsinki, Tampere) To engage in a thorough investigation of the new dependence logic, to find its axiomatization, its fine structure, and its relation to modal dependence theories.
- IMPINF (Aachen, Amsterdam, Paris) To develop a uniform logical and operational framework for handling imperfect information in logical games and other interactive systems.
- DYN (Oxford, Aachen, Amsterdam, Gothenburg) To relate and merge the two major existing approaches to the logic of interaction: local and global dynamics.
- LOGCON (Gothenburg, Amsterdam, Helsinki, Tampere) To apply existing methods from logic and mathematics for characterizing the standard logical constants for proof and truth to logical frameworks specifically designed to deal with interaction.

Idea: Interaction in a general model theoretic setting

- Hodges generalization of model theory to teams (or trumps) gives us lots of new quantifiers.
- Some of these new quantifiers should model interaction and games (in some way).
- Which of these are the interesting ones? Which are the logical ones?

Functional dependence

Dependence logic

Henkin / Hintikka / Sandu / Hodges / Väänänen

- Syntax of dependence logic: FOL + $(t_1, \ldots, t_{k-1} \rightsquigarrow t_k)$.
- Assume all formulas in negation normal form.
- Game theoretic semantics "as usual".
- Game of imperfect information (leading to three truth values).
- Semantics only works for sentences, thus not compositional.
- IF-logic $\approx$ Dependence logic.

Teams (or Trumps, or sets of assignments)

- Let $X \subseteq M^k$. $M \models_{\bar{x}/X} \varphi$ where the free variables of φ are among $\bar{x}$.
- Given $s \in X$ $\bar{x}/s$ assigns s_k to x_k.
- $M \models_{\bar{x}/X} P(\bar{t})$ iff $\forall s{\in}X, M \models_{\bar{x}/s} P(\bar{t})$
- $M \models_{\bar{x}/X} \neg P(\bar{t})$ iff $\forall s{\in}X, M \models_{\bar{x}/s} \neg P(\bar{t})$
- $M \models_{\bar{x}/X} \bar{t} \rightsquigarrow t'$ iff $\forall s, s'{\in}X(\forall i(t_i^{M,\bar{x}/s} = t_i^{M,\bar{x}/s'}) \Rightarrow t'^{M,\bar{x}/s} = t'^{M,\bar{x}/s'})$
- $M \models_{\bar{x}/X} \neg\bar{t} \rightsquigarrow t'$ iff $X = \emptyset$
- $M \models_{\bar{x}/X} \varphi \wedge \psi$ iff $M \models_{\bar{x}/X} \varphi$ and $M \models_{\bar{x}/X} \psi$
- $M \models_{\bar{x}/X} \varphi \vee \psi$ iff $\exists Y{\cup}Z{=}X, M \models_{\bar{x}/Y} \varphi$ and $M \models_{\bar{x}/Z} \psi$
- $M \models_{\bar{x}/X} \exists y\varphi$ iff $\exists F{:}X{\to}M, M \models_{\bar{x},y/X[F]} \varphi$
- $X[F] = \{\, s, F(s) \mid s \in X \,\}$
- $M \models_{\bar{x}/X} \forall y\varphi$ iff $M \models_{\bar{x},y/X[M]} \varphi$
- $X[M] = \{\, s, a \mid s \in X, a \in M \,\}$
- $M \models_X \varphi$

Some facts of DL

- For FOL-formulas φ we have $M \models_{\bar{x}/X} \varphi$ iff $M \models_{\bar{x}/s} \varphi$ for every $s \in X$.
- Empty team: $M \models_{\emptyset} \varphi$ for any φ.
- $\neg$ LEM: There are sentences σ such that $M \not\models \sigma \vee \neg\sigma$.
- Monotonicity: If $M \models_X \varphi$ and $Y \subseteq X$ then $M \models_Y \varphi$.
- Weakness: For sentences σ there is translation $\hat{\sigma}$ to Σ^1_1 such that $\sigma \equiv \hat{\sigma}$.
- Strength: For Σ^1_1 sentences Φ there is a translation $\hat{\Phi}$ to DL such that $\Phi \equiv \hat{\Phi}$.
- Thus, there is a sentence σ such that $(M, R) \models \sigma$ iff R is not well-founded on M. (However, being well-founded is not axiomatizable.)
- There is a formula $\mathrm{Tr}(x)$ of $\mathscr{L}_A$ such that $\mathrm{PA} \models \mathrm{Tr}(\sigma) \leftrightarrow \sigma$ for all sentences σ. (Remember that LEM is not valid.)

Invariance and logical constants

An operator (function/predicate) is a logical constant if it is **topic neutral**.

- Examples: $\exists$, $\forall$, $\neg$, and $\to$.
- Non-example: "for all even numbers"
- Debatable: "for infinitely many", $=$

Mautner, Tarski, Mostowski, Lindenbaum: Logic is the the study of the invariants under the most general transformations (=permutations). (cf. Klein's Erlangen program)

Definition (Lindström/Mostowski)

A (global) **generalized quantifier** *Q of type $\langle n_1, \ldots, n_k \rangle$ is a (class) of structures in the language $\{ R_1, \ldots, R_k \}$ where R_i is of arity n_i.*

Examples:

- $\exists = \{ (M, A) \mid A \subseteq M, A \neq \emptyset \}$
- $\forall = \{ (M, M) \mid M \}$
- $Q_0 = \{ (M, A) \mid A \subseteq M, |A| \geq \aleph_0 \}$
- $\exists^{=\kappa} = \{ (M, A) \mid A \subseteq M, |A| = \kappa \}$
- $I = \{ (M, A, B) \mid A, B \subseteq M, |A| = |B| \}$
- $W = \{ (M, R) \mid R \subseteq M^2, R \text{ is well-founded} \}$
- $Q^A = \{ (M, B) \mid A \subseteq B \}$

- $\varphi(M) = \{ \bar{a} \in M^k \mid M \models \varphi(\bar{a}) \}$
- $M \models Qx_0 \dots x_{k-1}\varphi(x_0, \dots, x_{k-1})$ *iff* $(M, \varphi(M)) \in Q$ *(Q of type $\langle k \rangle$)*

Local versions: For a given domain M, let (for Q of type $(\langle k \rangle)$

$$Q_M = \left\{ R \subseteq M^k \;\middle|\; (M, A) \in Q \right\}.$$

A (local) quantifier Q_M of type $\langle k \rangle$ is definable in the logic $\mathscr{L}$ if there is φ of $\mathscr{L}$, such that

$$(M, R) \models \varphi \text{ iff } R \in Q_M.$$

Tarski's thesis

A (local) quantifier on a domain M is a logical constant iff it is invariant under all **permutations** *of M.*

Examples: $\exists, \forall, Q_0, \exists^{=\kappa}, I$
Non-examples: Q^A

Mostowski's thesis

A quantifier Q is a logical constant iff it is invariant under all **bijections** *(across domains).*

Theorem (McGee -91 / Krasner -38)

Q is bijection invariant iff for each κ there is a formula in $\mathscr{L}_{\infty\infty}$ defining Q_M for each $|M| = \kappa$.

Fix a domain Ω. Quantifier means local quantifier on Ω.
$\mathcal{Q}$ is a set of quantifiers.
G subgroup of the full symmetric group S_Ω.

Definition

- *Let* $\mathrm{Aut}(\mathcal{Q})$ *be the group of all permutations of* Ω *fixing all quantifiers in* $\mathcal{Q}$*:*
  $\mathrm{Aut}(\mathcal{Q}) = \{\, g \in S_\Omega \mid g(Q) = Q \text{ for all } Q \in \mathcal{Q} \,\}$.
- *Let* $\mathrm{Inv}(G)$ *be the set of quantifiers fixed by* G*:*
  $\mathrm{Inv}(G) = \{\, Q \mid g(Q) = Q \text{ for all } g \in G \,\}$.

Theorem (Krasner/Bonnay/E)

- $\mathrm{Aut}(\mathrm{Inv}(G)) = G$
- $\mathrm{Inv}(\mathrm{Aut}(\mathcal{Q}))$ *is the set of quantifiers definable in* $\mathscr{L}_{\infty\infty}(\mathcal{Q})$

Proof

$\mathrm{Aut}(\mathrm{Inv}(G)) = G$: Let $\leq$ well-order Ω, and $Q = \{ g(\leq) \mid g \in G \}$ of type $\langle 2 \rangle$. If $h \in \mathrm{Aut}(\mathrm{Inv}(G))$ then $h(\leq) \in Q$ and so there is $g \in G$ such that $h(\leq) = g(\leq)$, implying $h = g$.

$\mathrm{Inv}(\mathrm{Aut}(\mathscr{Q}))$ is the set of Qs definable in $\mathscr{L}_{\infty\infty}(\mathscr{Q})$: We assume all quantifiers of type $\langle 1 \rangle$ and $\Omega = \omega$.
$Q' \in \mathrm{Inv}(\mathrm{Aut}(\mathscr{Q}))$ is defined by

$$\forall x_0, x_1, \ldots \Big[\bigwedge_{i \neq j} x_i \neq x_j \wedge \forall y \bigvee_i y = x_i \wedge \\ \bigwedge_{Q \in \mathscr{Q}} \Big(\big(\bigwedge_{A \in Q} Qy \bigvee_{i \in A} y = x_i \big) \wedge \big(\bigwedge_{A \notin Q} \neg Qy \bigvee_{i \in A} y = x_i \big) \Big) \rightarrow \\ \bigvee_{A \in Q'} \Big(\bigwedge_{i \in A} Px_i \wedge \bigwedge_{i \notin A} \neg Px_i \Big) \Big]$$

Theorem

If $\mathrm{Inv}_m(G)$ *are all* **monadic** *quantifiers invariant under* G *then there is a subgroup* G *such that* $\mathrm{Aut}(\mathrm{Inv}_m(G)) \supsetneq G$.

Proof. Let G be the group of **piecewise monotone** permutations on ω: $g \in S_\omega$ is piecewise monotone if there exists partitions $A_1 \cup \ldots \cup A_k = B_1 \cup \ldots \cup B_k = \omega$ such that $g|A_i$ is the unique increasing function $A_i \to B_i$.
$\mathrm{Aut}(\mathrm{Inv}_m(G))$ is closed in the topology generated by

$$U_{\bar{A},\bar{B}} = \{ h \in S_\omega \mid h(A_i) = B_i \text{ all } i < k \}$$

as basic open sets, where $\bar{A} = A_0, \ldots, A_{k-1}$ and $\bar{B} = B_0, \ldots, B_{k-1}$ are subsets of ω.
The closure of G is S_ω.

Feferman's thesis -99

Definition

A (global) quantifier Q is invariant under **preimages of surjections** *if for every $h : M \to N$ surjection and for all $R \subseteq N^k$: $h^{-1}(R) \in Q_M$ iff $R \in Q_N$.*

Theorem (Feferman)

Quantifiers of type $\langle 1, \ldots, 1 \rangle$ are invariant under **preimages of surjections** *iff they are definable in $\mathscr{L}^-_{\omega\omega}$.*

Feferman's thesis

A quantifier is a logical constant iff it can de defined (in typed λ-calculus) from equality and monadic quantifiers invariant under preimages of surjections.

$h : M \to N$ can be "lifted" by: $h(Q_M) = \{\, h(R) \mid R \in Q_M \,\}$.
Invariance under **surjections**: $h(Q_M) = Q_N$ for all surjective h.

Theorem (Casanovas -07)

"Quantifiers" are invariant under surjections iff they are definable in a certain positive fragment of $\mathscr{L}_{\omega\omega}$ (with restricted use of equality).

Invariance under **back-and-forth equivalence**: If (M, A) and (N, B) are back-and-forth equivalent, then $A \in Q_M$ iff $B \in Q_N$.

Theorem (Barwise -73)

A local quantifier Q on M is **back-and-forth invariant** *iff Q is definable in $\mathscr{L}_{\infty\omega}$.*

Invariance and teams

The "classical" case

- Fix a model M.
- $[\bar{x}|\varphi] = \left\{ \bar{a} \in M \;\middle|\; M \models_{\bar{x}/\bar{a}} \varphi \right\} \in \mathscr{P}(M^k)$
- Observe that $[\epsilon|\sigma]$ is $T = \{ \epsilon \}$ or $F = \emptyset$.
- A $\langle k_1, \ldots, k_l \rangle$-quantifier (on M) is a subset of $\mathscr{P}(M^{k_1}) \times \ldots \times \mathscr{P}(M^{k_l})$.

DL-semantics revisited

- Let $[\bar{x}|\varphi]_M = \{\, X \mid M \models_{\bar{x}/X} \varphi \,\}$
- $[\bar{x}|\varphi \wedge \psi] = [\bar{x}|\varphi] \cap [\bar{x}|\psi]$
- $[\bar{x}|\varphi \vee \psi] = \{\, Y \cup Z \mid Y \in [\bar{x}|\varphi], Z \in [\bar{x}|\psi] \,\}$
- $[\bar{x}|\exists y\varphi] = \{\, X \mid \exists F : X \to M, X[F] \in [\bar{x}, y|\varphi] \,\}$
- $[\bar{x}|\forall y\varphi] = \{\, X \mid X[M] \in [\bar{x}, y|\varphi] \,\}$
- The set $[\varphi]$ is an order ideal in the po-set $(\mathscr{P}(M^k), \subseteq)$.
- $\emptyset \in [\varphi]$.

The semantic setup

- $T_k = \mathscr{P}(M^k)$
- $S_k = \{\, \mathscr{X} \subseteq T_k \mid \mathscr{X} \text{ order ideal} \,\}$. ($\mathscr{X}$ order ideal if $Y \subseteq X \in \mathscr{X}$ implies $Y \in \mathscr{X}$.)
- $S_0 = \{\, \emptyset = \bot, \{\, \emptyset \,\} = 0, \{\, \emptyset, \{\, \epsilon \,\} \,\} = \top \,\}$.
- To get a two-valued logic we could demand all the ideals in S_k to include $\emptyset$. (That would not make $\multimap$ fit in the framework.)
- By easing on the definition of S_k we could make "real" negation part of the framework, however the analysis would be much more cumbersome.

Quantifers

- A $\langle k_1, \ldots, k_l \rangle$ (local) quantifier $\mathscr{Q}$ is a function $\mathscr{Q} : S_{k_1} \times \ldots \times S_{k_l} \to S_0$.
- A $\langle k_1, \ldots, k_l \rangle$ (local) operator Φ, gives for each n a function $\Phi_n : S_{n+k_1} \times \ldots \times S_{n+k_l} \to S_n$.
- We say that a quantifier $\mathscr{Q}$ (operator Φ) is two-valued if for every $\emptyset \in \mathscr{X}$ we have $\emptyset \in \mathscr{Q}\mathscr{X}$ ($\emptyset \in \Phi\mathscr{X}$).

Examples of quantifiers/operators

- $\exists(\mathscr{X}) = \{\, X \in T_k \mid \exists F : X \to M, X[F] \in \mathscr{X} \,\}$
- $\forall(\mathscr{X}) = \{\, X \in T_k \mid X[M] \in \mathscr{X} \,\}$
- $\exists^1(\mathscr{X}) = \{\, X \in T_k \mid X[a] \in \mathscr{X}$ for some $a \in M \,\}$
- $\forall^1(\mathscr{X}) = \{\, X \in T_k \mid X[a] \in \mathscr{X}$ for all $a \in M \,\}$
- $\exists_0 = \exists^1_0$
- $\forall_0 \neq \forall^1_0$

Connectives

A connective is an operator of type $\langle 0, \ldots, 0 \rangle$.

- $\mathscr{X} \vee \mathscr{Y} = \mathscr{X} \cup \mathscr{Y}$
- $\mathscr{X} \wedge \mathscr{Y} = \mathscr{X} \cap \mathscr{Y}$
- $\mathscr{X} \otimes \mathscr{Y} = \{ X \cup Y \mid X \in \mathscr{X}, Y \in \mathscr{Y} \}$
- $\mathscr{X} \oplus \mathscr{Y} = \{ Z \mid \forall X \cup Y = Z, X \in \mathscr{X}, Y \in \mathscr{Y} \}$
- $\mathscr{X} \rightarrow \mathscr{Y} = \{ Z \mid \mathscr{P}(Z) \cap \mathscr{X} \subseteq \mathscr{Y} \}$
- $\mathscr{X} \multimap \mathscr{Y} = \{ Z \mid \forall X \in \mathscr{X}, Z \cup X \in \mathscr{Y} \}$

Non-example: $\sim \mathscr{X} = \{ X \mid X \notin \mathscr{X} \}$
Not truth-functions.

Negation

In this setting there is no way of treating (game theoretic) negation as a quantifier/operator: Assume $|M| > 1$. Let σ be $\forall x(\epsilon \rightsquigarrow x)$ then

$$[\sigma] = \{ \emptyset \}$$

$$[\neg\sigma] = [\exists x \neg(\epsilon \rightsquigarrow x)] = \{ \emptyset \}$$

Let τ be $\exists x(x \neq x)$ then

$$[\tau] = \{ \emptyset \}$$

$$[\neg\tau] = [\forall x(x = x)] = \{ \emptyset, \{ \epsilon \} \}$$

To make negation compositional in this sense it seems necessary to complicate the semantic framework considerably.

Atoms

- k-ary atoms $\leftrightarrow$ first-order $\langle k \rangle$-quantifiers.
- $=(R)$ iff $R \subseteq =$
- $\rightsquigarrow (R)$ iff $R \subseteq \mathrm{graph}(f)$ for some function f.

What are the right notions of invariance and definability to get theorems as in the first-order case?

Lunch