Omitting types in expansions and related strong saturation properties

Work in progress

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These slides are available at:

`http://www.math.chalmers.se/~engstrom`.

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- We write $p{\uparrow}$ for the $\mathscr{L}^+_{\omega_1\omega}$-sentence

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- We look at properties of the kind:

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- For first-order theories these are the same, but not for infinitary logics.

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- Given a Scott set $\mathfrak{X}$, $\mathfrak{M}$ is $\mathfrak{X}$-saturated if

$$\mathfrak{M} \models p(\bar{x}, \bar{a}){\downarrow} \quad \text{iff} \quad p(\bar{x}, \bar{y}) \in \mathfrak{X}$$

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- Every recursively saturated model $\mathfrak{M}$ is $\mathfrak{X}$-saturated for some Scott set $\mathfrak{X}$.
- Every $\mathfrak{X}$-saturated model is recursively saturated.

Resplendency

- A model $\mathfrak{M}$ is *resplendent* if for every $\bar{a} \in \mathfrak{M}$, every recursive extension $\mathscr{L}^+$ of $\mathscr{L}(\bar{a})$ and every recursive T in $\mathscr{L}^+$ such that $\mathrm{Th}(\mathfrak{M}, \bar{a}) + T$ is consistent there is an expansion $\mathfrak{M}^+$ of $\mathfrak{M}$ satisfying T.

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- Every countable recursively saturated model is resplendent, and every resplendent model is recursively saturated.

Subresplendency

- A model $\mathfrak{M}$ is *subresplendent* if for every $\bar{a} \in \mathfrak{M}$, every recursive extension $\mathscr{L}^+$ of $\mathscr{L}(\bar{a})$ and every recursive T in $\mathscr{L}^+$ such that $\mathrm{Th}(\mathfrak{M}, \bar{a}) + T$ is consistent there is an elementary submodel $\bar{a} \in \mathfrak{N} \prec M$ and an expansion $\mathfrak{N}^+$ of $\mathfrak{N}$ satisfying T.

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- A model is subresplendent iff it is recursively saturated.

A strong saturation property

- A model which is $\mathfrak{X}$-saturated for a Scott set $\mathfrak{X}$ such that if $\bar{a} \in \mathfrak{M}$, $T, p \in \mathfrak{X}$ and $\omega\text{--}\mathrm{SatCon}_{\mathscr{L}}(\mathrm{Th}(\mathfrak{M}, \bar{a}) + T + p\!\uparrow)$ then there exists a completion $T^c \in \mathfrak{X}$ of T such that $\omega\text{--}\mathrm{SatCon}_{\mathscr{L}}(\mathrm{Th}(\mathfrak{M}, \bar{a}) + T^c + p\!\uparrow)$ is called p-saturated.

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- Every p-saturated countable model $\mathfrak{M}$ is p-resplendent:

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- A model is *analytically saturated* if it is $\mathfrak{X}$-saturated for a Scott set $\mathfrak{X}$ which satisfies all *true* second-order arithmetic sentences with parameters from $\mathfrak{X}$.
- If $\mathfrak{M} \models \mathrm{PA}$ is p-resplendent then it is analytically saturated.
- Are analytic saturation and p-saturation equivalent for models of PA?

β-saturation

- A model is β-*saturated* if it is $\mathfrak{X}$-saturated for a Scott set $\mathfrak{X}$ which satisfies all true Σ^1_1 arithemtic sentences with parameters from $\mathfrak{X}$ (i.e., $\mathfrak{X}$ is a β-model).

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- Every β-saturated model $\mathfrak{M}$ is p-subresplendent:

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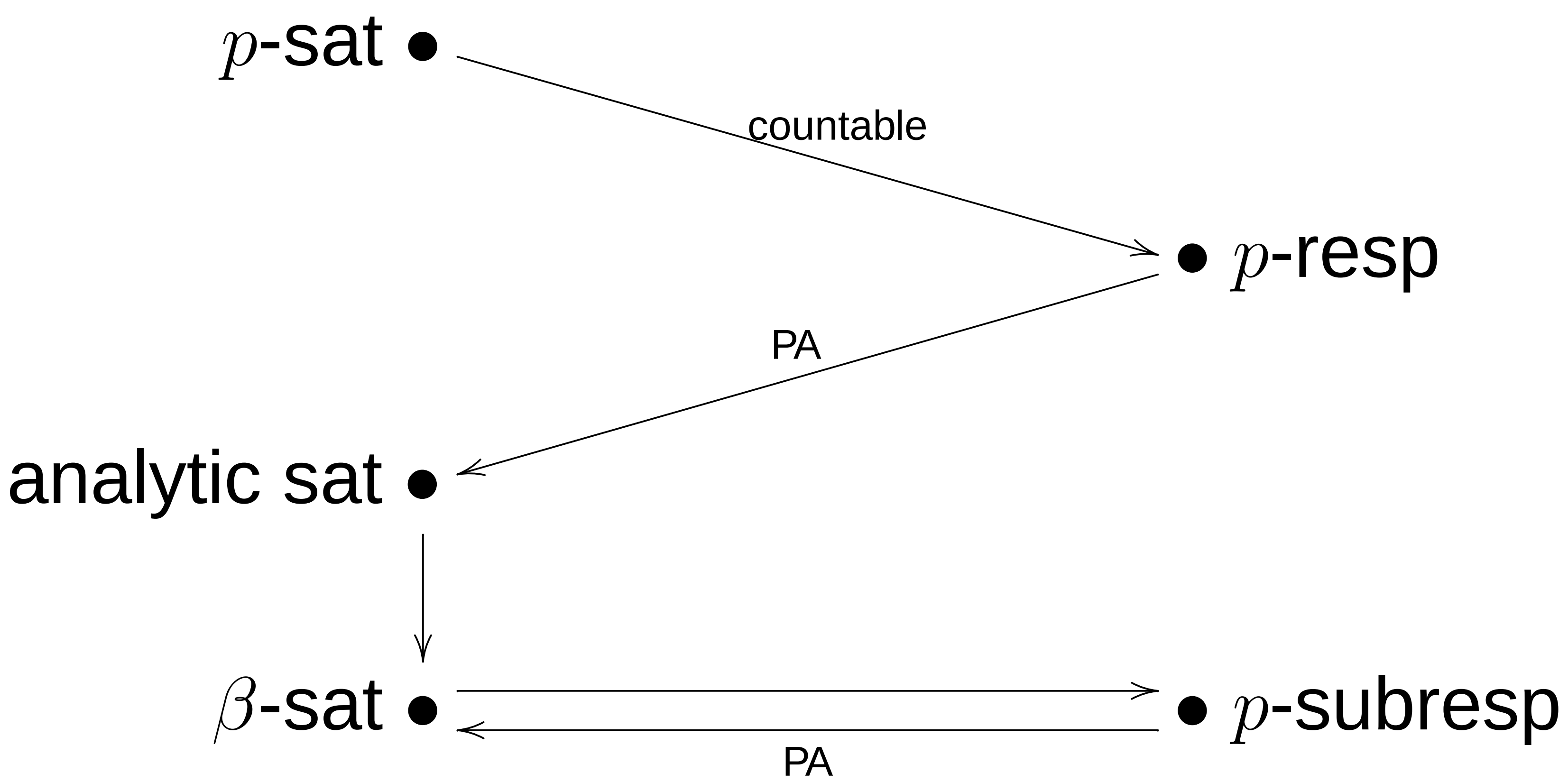
Summary



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- A theory T (in an infinitary logic) is called *pervasive* over $\mathfrak{M}$ if for any $\mathfrak{N} \succ \mathfrak{M}$ there is an $\mathfrak{K} \succ \mathfrak{N}$ and an expansion $\mathfrak{K}^+$ of $\mathfrak{K}$ such that $\mathfrak{K}^+ \models T$.

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- Replace $\omega\text{–}\mathrm{SatCon}_{\mathscr{L}}(\mathrm{Th}(\mathfrak{M}, \bar{a}) + T + p\!\uparrow)$ with $T + p\!\uparrow$ is pervasive over $\mathfrak{M}$.