Invariance and definability, with or without equality

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Introduction

Invariance

- **Klein's Erlangen Program:** Invariance as the defining property for geometries.
- **Tarski's thesis:** Extend to logics; use invariance as defining property for logics and logical operators. (Tarski, 1986)
- **Idea**: Extend the correspondence of invariance and operators to a full Galois connection: **Inv** maps invariance criteria to sets of operators, and **Aut** maps sets of operators to invariance critera such that

$$\mathscr{Q} \subseteq \mathrm{Inv}(H) \text{ iff } H \subseteq \mathrm{Aut}(\mathscr{Q}), \text{ and}$$

$\mathrm{Inv}(\mathrm{Aut}(\mathscr{Q}))$ corresponds to definability in a logic L.

QUANTIFIERS

DEFINITION (MOSTOWSKI/LINDSTRÖM)

A **generalized quantifier** Q of type $\langle n_1, \ldots, n_k \rangle$ is a (class) of structures in the language $\{ \mathtt{R}_1, \ldots, \mathtt{R}_k \}$ where $\mathtt{R}_i$ is of arity n_i.

Examples:

- $\exists = \{ (M, A) \mid A \subseteq M, A \neq \emptyset \}$
- $\forall = \{ (M, M) \mid M \}$
- $Q_0 = \{ (M, A) \mid A \subseteq M, |A| \geq \aleph_0 \}$

DEFINITION

$M \models Q\bar{x}\,\varphi(\bar{x})$ iff $(M, R) \in Q$, where $R = \left\{ \bar{a} \in \Omega^k \;\middle|\; M \models \varphi(\bar{a}) \right\}$.

- **Local** quantifier: $Q_\Omega = \{ \langle R_1, \ldots, R_k \rangle \mid (\Omega, R_1, \ldots, R_k) \in Q \}$
- A local quantifier, of type $\langle n \rangle$, is definable on Ω in the logic $\mathscr{L}$ if there is φ of $\mathscr{L}$, such that $(\Omega, R) \models \varphi$ iff $R \in Q$.

Galois theory:

$$\{\, H \subseteq \mathrm{Aut}(K:k) \,\} \rightleftharpoons \{\, A \mid k \subseteq A \subseteq K \,\}$$

least group ⇌ least field

Krasner's Galois theory:

$$\{\, H \subseteq \mathrm{Sym}(\Omega) \,\} \rightleftharpoons \{\, M \text{ infinitary rel. structure on } \Omega \,\}$$

least group ⇌ definability in $\mathscr{L}_{\infty\infty}$

Our results:

$$\{\, H \subseteq \mathrm{Sym}(\Omega) \,\} \rightleftharpoons \{\, \mathscr{Q} \text{ set of quantifiers on } \Omega \,\}$$

least group ⇌ definability in $\mathscr{L}_{\infty\infty}$

$$\{\, \Pi \text{ set of similarities on } \Omega \,\} \rightleftharpoons \{\, \mathscr{Q} \text{ set of quantifiers on } \Omega \,\}$$

least full monoid ⇌ definability* in $\mathscr{L}^{-}_{\infty\infty}$

Motivation I

Tarski's thesis on logicality (Tarski, 1986)

A (local) quantifier on a domain Ω is a logical constant iff it is invariant under all **permutations** of Ω.

McGee's Theorem (McGee, 1996)

A local quantifier Q on Ω is permutation invariant iff it is $\mathscr{L}_{\infty\infty}$-definable.

- Galois connection results give stronger connections between logics and invariance criteria: The connections are **stable** under adding operations.

Motivation II

Monadic quantifier: Quantifiers of type $\langle 1, \dots, 1 \rangle$.

Feferman's thesis on logicality (Feferman, 1999)

A quantifier is a logical constant iff it can de defined (in typed λ-calculus) from equality and **monadic** quantifiers invariant under talking preimages of surjections.

Feferman's Theorem (Feferman, 1999)

Monadic quantifiers are invariant under **preimages of surjections** iff they are definable in $\mathscr{L}^{-}_{\omega\omega}$.

- Feferman leaves the general question for arbitrary quantifiers open.
- Our result on the equality-free version of $\mathscr{L}_{\infty\infty}$ is a variant on Feferman's theorem, generalized to a full Galois connection.

With equality

A Galois connection

- Fix a domain Ω. Quantifier means local quantifier on Ω.
- $\mathcal{Q}$ is a set of quantifiers.
- G subgroup of the full symmetric group $\mathrm{Sym}(\Omega)$.

Definition

- Let **Aut**$(\mathcal{Q})$ be the group of all permutations of Ω fixing all quantifiers in $\mathcal{Q}$:
  $\mathrm{Aut}(\mathcal{Q}) = \{\, g \in \mathrm{Sym}(\Omega) \mid g(Q) = Q \text{ for all } Q \in \mathcal{Q} \,\}$.
- Let **Inv**(G) be the set of quantifiers fixed by G:
  $\mathrm{Inv}(G) = \{\, Q \mid g(Q) = Q \text{ for all } g \in G \,\}$.

Theorem (Krasner, 1938, 1950), (B/E)

- $\mathrm{Aut}(\mathrm{Inv}(G)) = G$
- $\mathrm{Inv}(\mathrm{Aut}(\mathcal{Q}))$ is the set of quantifiers definable in $\mathcal{L}_{\infty\infty}(\mathcal{Q})$

There is a permutation group which is not $\mathrm{Aut}(\mathcal{Q})$ for any set of monodic quantifiers $\mathcal{Q}$.

Proof

$\mathrm{Aut}(\mathrm{Inv}(G)) = G$: Let $\leq$ well-order Ω, and $Q = \{\, g(\leq) \mid g \in G \,\}$ of type $\langle 2 \rangle$. If $h \in \mathrm{Aut}(\mathrm{Inv}(G))$ then $h(\leq) \in Q$ and so there is $g \in G$ such that $h(\leq) = g(\leq)$, implying $h = g$.

$\mathrm{Inv}(\mathrm{Aut}(\mathscr{Q}))$ is the set of Qs definable in $\mathscr{L}_{\infty\infty}(\mathscr{Q})$: We assume all quantifiers of type $\langle 1 \rangle$ and $\Omega = \omega$.
$Q' \in \mathrm{Inv}(\mathrm{Aut}(\mathscr{Q}))$ is defined by

$$\forall x_0, x_1, \ldots \Big[\bigwedge_{i \neq j} x_i {\neq} x_j \wedge \forall y \bigvee_i y{=}x_i \wedge \bigwedge_{Q \in \mathscr{Q}} \Big(\big(\bigwedge_{A \in Q} Qy \bigvee_{i \in A} y{=}x_i \big) \wedge \big(\bigwedge_{A \notin Q} \neg Qy \bigvee_{i \in A} y{=}x_i \big) \Big) \to \bigvee_{A \in Q'} \Big(\bigwedge_{i \in A} Px_i \wedge \bigwedge_{i \notin A} \neg Px_i \Big) \Big]$$

Without equality

Plan

- We want a Galois connection involving the equality free logic $\mathscr{L}^{-}_{\infty\infty}$.
- **Idea:** Work in $\Omega/\sim$, where $\sim$ is the finest definable equivalance relation and apply the previous result.
- **Problem:** Can we define $\sim$ without knowing the language?
- **Solution:** Yes... sometimes.

Definitions

- π is a **similarity relation** on Ω if $\mathrm{dom}(\pi) = \mathrm{rng}(\pi) = \Omega$.
- $R\ \pi\ S$ if $\forall \bar{a}, \bar{b} \in \Omega$ such that $\bar{a}\ \pi\ \bar{b}$: $\bar{a} \in R$ iff $\bar{b} \in S$.
- R is **invariant** under π if $R\ \pi\ R$.

Invariance for quantifiers is parametrized by an equivalence relation:

Definition

A quantifier Q on Ω is **$\sim$-invariant** under π if for all relations R_1, ..., R_k, S_1, ..., S_k on Ω **invariant under** $\sim$ such that $R_i\ \pi\ S_i$ we have $\langle R_1, \ldots R_k \rangle \in Q$ iff $\langle S_1, \ldots, S_k \rangle \in Q$.

Motivation: The language $\mathscr{L}^{-}_{\infty\infty}(\mathscr{Q})$ can be very restricted: we can only talk about the **definable** sets/relations.

The mappings

- A set of operations $\mathscr{Q}$ generates an equivalence relation $\sim_{\mathscr{Q}}$, the finest $\mathscr{L}^{-}_{\infty\infty}(\mathscr{Q})$-definable equivalence relation.
- Dually, a set of similarities Π gives us an equivalence relation by the following condition:

  $a \approx_{\Pi} b$ if for all $\bar{c} \in \Omega^k$ there is $\pi \in \Pi$ such that $a, \bar{c}\ \pi\ b, \bar{c}$.

The mappings for the Galois connection can now be defined:

- **Sim**$(\mathscr{Q})$ is the set of similarities π such that all relations and quantifiers in $\mathscr{Q}$ are $\sim_{\mathscr{Q}}$-invariant under π.
- **Inv**(Π) is the set of all relations R and quantifiers Q on Ω which are $\approx_{\Pi}$-invariant under all similarities in Π.

First half of the correpsondence

Let the **blow-up** $\hat{Q}$ of Q relative to $\sim$ be $\{\hat{R} \mid R \in Q\}$, where

$$\hat{R} = \{ \langle a_1, \ldots, a_k \rangle \mid \exists \langle b_1, \ldots, b_k \rangle \in R, a_1 \sim b_1, \ldots a_k \sim b_k \} .$$

Theorem

Let $\mathscr{Q}$ be a set of operators then

1. $Q \in \mathrm{Inv}(\mathrm{Sim}(\mathscr{Q}))$ iff $\hat{Q}$ is definable in $\mathscr{L}^{-}_{\infty\infty}(\mathscr{Q})$.
2. $R \in \mathrm{Inv}(\mathrm{Sim}(\mathscr{Q}))$ iff R is definable in $\mathscr{L}^{-}_{\infty\infty}(\mathscr{Q})$.

More definitions

- A similarity π is **identity-like** (with respect to Π) if $\pi \subseteq \approx_{\Pi}$.
- A set Π of similarties is **saturated** if it includes all identity-like similarities.
- Π is a **monoid with involution** if it is closed under composition and taking converses.
- Π is **full** if it is a saturated monoid with involution closed under taking subsimilarities, i.e., such that if $\pi \in \Pi$ and $\pi' \subseteq \pi$ is a similarity then $\pi' \in \Pi$.

Theorem

Let Π be a set of similarity relations, then

$$\mathrm{Sim}(\mathrm{Inv}(\Pi)) \text{ is the smallest full monoid including } \Pi.$$

Thank you for your attention.

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