On logicality

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Introduction

Logicality and validity

Logic considers the **form** of sentences and arguments. To determine this form we need to know what the **logical constants** are.

Which of the symbols/expressions should be considered **logical**?

Once a demarcation is made Bolzano's analysis of logical consequence makes sense:

Bolzano
An argument is **logically valid** if no reinterpretation/substitution of its non-logical expressions makes the premises true and the conclusion false.

A model theoretic approach

Ryle (1954)
An operator (function/predicate) is a logical constant if it is **topic neutral**.

Mautner (1946); Tarski (1986)
Logic is the the study of the invariants under the most general transformations.

Compare with Klein's Erlangen program for classifying geometries in terms of invariance.

Quantifiers

Definition Lindström (1966); Mostowski (1957)

- A (global) **generalized quantifier** Q of type $\langle n_1, \ldots, n_k \rangle$ is a (class) of structures in the language $\{ \mathrm{R}_1, \ldots, \mathrm{R}_k \}$ where R_i is of arity n_i.
- $M \models_s Q\bar{x}_1, \ldots, \bar{x}_k(\varphi_1, \ldots, \varphi_k)$ iff $(M, \varphi_1^{M,s}, \ldots, \varphi_k^{M,s}) \in Q$.

Examples:

- $\exists = \{ (M, A) \mid A \subseteq M, A \neq \emptyset \}$
- $\forall = \{ (M, M) \mid \top \}$
- $Q_0 = \{ (M, A) \mid A \subseteq M, |A| \geq \aleph_0 \}$
- $I = \{ (M, A, B) \mid \ |A| = |B| \}$

A quantifier Q is **definable in the logic $\mathscr{L}$** if there is φ of $\mathscr{L}(\mathrm{R}_1, \ldots, \mathrm{R}_k)$, such that

$$(M, R_1, \ldots, R_k) \models \varphi \text{ iff } (M, R_1, \ldots, R_k) \in Q.$$

Definition
A **local quantifier** on the domain M is a set of the form

$$Q_M = \{ (R_1, \ldots, R_k) \mid (M, R_1, \ldots, R_k) \in Q \}$$

for some generalized quantifier Q.

Tarski's thesis
A (local) quantifier on a domain M is a logical constant iff it is invariant under all **permutations** of M.

Mostowski's thesis
A quantifier Q is a logical constant iff it is invariant under all **bijections** (across domains).

Theorem (McGee (1996); Krasner (1938))
Q is bijection invariant iff for each κ there is a formula in $\mathscr{L}_{\infty\infty}$ defining Q_κ.

Alternative notions of invariance I

Definition
A (global) quantifier Q is invariant under **preimages of surjections** if for every $h : M \to N$ surjection and for all $R \subseteq N^k$: $h^{-1}(R) \in Q_M$ iff $R \in Q_N$.

Theorem (Feferman)
Quantifiers of type $\langle 1, \ldots, 1 \rangle$ are invariant under **preimages of surjections** iff they are definable in $\mathscr{L}^-_{\omega\omega}$.

Feferman's (old?) thesis -99
A quantifier is a logical constant iff it can de defined (in typed λ-calculus) from equality and monadic quantifiers invariant under preimages of surjections.

Alternative notions of invariance II

$h : M \to N$ can be "lifted" by: $h(Q_M) = \{ h(R) \mid R \in Q_M \}$.

- Invariance under **surjections**: $h(Q_M) = Q_N$ for all surjective h.

Theorem (Casanovas, 2007)
Quantifiers are invariant under surjections iff they are definable in a certain fragment of $\mathscr{L}_{\omega\omega}$.

- Invariance under **back-and-forth equivalence**: If (M, A) and (N, B) are back-and-forth equivalent, then $A \in Q_M$ iff $B \in Q_N$.

Theorem (Barwise, 1973)
A local quantifier Q on M is **back-and-forth invariant** iff Q is definable in $\mathscr{L}_{\infty\omega}$.

Galois connections

Invariance

- **Klein's Erlangen Program:** Invariance as the defining property for geometries.
- **Tarski's thesis:** Extend to logics; use invariance as defining property for logics and logical operators. (Tarski, 1986)
- **Idea**: Extend the correspondence of invariance and operators to a (antitone) Galois connection: **Inv** maps invariance criteria to sets of operators, and **Aut** maps sets of operators to invariance critera such that

$$q \subseteq \mathrm{Inv}(G) \text{ iff } G \subseteq \mathrm{Aut}(q).$$

- Also, we want $\mathrm{Inv}(\mathrm{Aut}(q))$ to correspond to definability in $\mathscr{L}(q)$ for some logic $\mathscr{L}$.

Krasner's Galois theory:

$\{ G \subseteq \mathrm{Sym}(\Omega) \}$ least group $\rightleftharpoons$ $\{ M$ infinitary rel. structure on $\Omega \}$ definability in $\mathscr{L}_{\infty\infty}$

Our results:

$\{ G \subseteq \mathrm{Sym}(\Omega) \}$ least group $\rightleftharpoons$ $\{ q$ set of quantifiers on $\Omega \}$ definability in $\mathscr{L}_{\infty\infty}$

$\{ \Pi$ set of similarities on $\Omega \}$ least full monoid $\rightleftharpoons$ $\{ q$ set of quantifiers on $\Omega \}$ $\hat{\ }$-definability in $\mathscr{L}^{-}_{\infty\infty}$

Motivation

- Galois connection results give stronger correspondences between logics and invariance criteria: They are **stable** under adding operations.

Feferman's Theorem (Feferman, 1999)
Monadic quantifiers are invariant under **preimages of surjections** iff they are definable in $\mathscr{L}^{-}_{\omega\omega}$.

- Feferman leaves the general question for arbitrary quantifiers open.
- Our result on the equality-free version of $\mathscr{L}_{\infty\infty}$ is a variant on Feferman's theorem, generalized to a full Galois connection.

With equality

A Galois connection

- Fix a domain Ω. Quantifier means local quantifier on Ω.
- q is a set of quantifiers.
- G subgroup of the full symmetric group $\mathrm{Sym}(\Omega)$.

Definition

- Let **Aut(q)** be the group of all permutations of Ω fixing all quantifiers in q: $\mathrm{Aut}(q) = \{ g \in \mathrm{Sym}(\Omega) \mid g(Q) = Q \text{ for all } Q \in q \}$.
- Let **Inv(G)** be the set of quantifiers fixed by G: $\mathrm{Inv}(G) = \{ Q \mid g(Q) = Q \text{ for all } g \in G \}$.

Theorem (Krasner, 1938, 1950), (B/E)

- $\mathrm{Aut}(\mathrm{Inv}(G)) = G$
- $\mathrm{Inv}(\mathrm{Aut}(q))$ is the set of quantifiers definable in $\mathscr{L}_{\infty\infty}(q)$

Proof

$\mathrm{Aut}(\mathrm{Inv}(G)) = G$: Let $\leq$ well-order Ω, and $Q = \{ g(\leq) \mid g \in G \}$ of type $\langle 2 \rangle$. If $h \in \mathrm{Aut}(\mathrm{Inv}(G))$ then $h(\leq) \in Q$ and so there is $g \in G$ such that $h(\leq) = g(\leq)$, implying $h = g$.

$\mathrm{Inv}(\mathrm{Aut}(q))$ is the set of Qs definable in $\mathscr{L}_{\infty\infty}(q)$: We assume all quantifiers of type $\langle 1 \rangle$ and $\Omega = \omega$.
$Q' \in \mathrm{Inv}(\mathrm{Aut}(q))$ is defined by

$$\forall x_0, x_1, \ldots \Big[\bigwedge_{i \neq j} x_i \neq x_j \wedge \forall y \bigvee_i y = x_i \wedge \bigwedge_{Q \in q} \Big(\big(\bigwedge_{A \in Q} Qy \bigvee_{i \in A} y = x_i \big) \wedge \big(\bigwedge_{A \notin Q} \neg Qy \bigvee_{i \in A} y = x_i \big) \Big) \rightarrow \bigvee_{A \in Q'} \Big(\bigwedge_{i \in A} Px_i \wedge \bigwedge_{i \notin A} \neg Px_i \Big) \Big]$$

Theorem
If $\mathrm{Inv}_m(G)$ are all **monadic** quantifiers invariant under G then there is a subgroup G such that $\mathrm{Aut}(\mathrm{Inv}_m(G)) \supsetneq G$.

Proof. Let G be the group of **piecewise monotone** permutations on ω: $g \in S_\omega$ is piecewise monotone if there exists partitions $A_1 \cup \ldots \cup A_k = B_1 \cup \ldots \cup B_k = \omega$ such that $g|A_i$ is the unique increasing function $A_i \to B_i$.
$\mathrm{Aut}(\mathrm{Inv}_m(G))$ is closed in the topology generated by

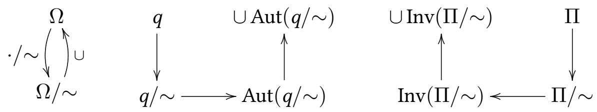
$$U_{\bar{A},\bar{B}} = \{ h \in \mathrm{Sym}(\omega) \mid h(A_i) = B_i \text{ all } i < k \}$$

as basic open sets, where $\bar{A} = A_0, \ldots, A_{k-1}$ and $\bar{B} = B_0, \ldots, B_{k-1}$ are subsets of ω.
The closure of G is $\mathrm{Sym}(\omega)$.

Without equality

Plan

- We want a Galois connection in which the closure operator on sets of quantifiers is definability in $\mathscr{L}^{-}_{\infty\infty}$.
- Idea: Work in $\Omega/\sim$, where $\sim$ is the finest definable equivalence relation and apply the previous result.



- Problem: Can we define $\sim$ without knowing the language?
- Solution: Yes... sometimes.

Similarity relations

- π is a similarity relation on Ω if $\mathrm{dom}(\pi) = \mathrm{rng}(\pi) = \Omega$.
- Every surjection is a similarity relation.
- For every similarity π there are surjections $f\colon \Omega \to \Omega'$ such that $\pi = f \circ g^{-1}$.
- $R\ \pi\ S$ if $\forall \bar{a}, \bar{b} \in \Omega$ such that $\bar{a}\ \pi\ \bar{b}$: $\bar{a} \in R$ iff $\bar{b} \in S$.
- R is invariant under π if $R\ \pi\ R$.

Invariance

Invariance for quantifiers is parametrized by an equivalence relation:

Definition
A quantifier Q on Ω is $\sim$-invariant under π if for all relations $R_1, \ldots, R_k$, $S_1, \ldots, S_k$ on Ω invariant under $\sim$ such that $R_i\ \pi\ S_i$ we have $\langle R_1, \ldots R_k\rangle \in Q$ iff $\langle S_1, \ldots, S_k\rangle \in Q$.

The mappings

- A set of operations q generates an equivalence relation $\sim_q$, the finest $\mathscr{L}^{-}_{\infty\infty}(q)$-definable equivalence relation.
- Dually, a set of similarities Π gives us an equivalence relation by the following condition:
  $a \approx_\Pi b$ if for all $\bar{c} \in \Omega^k$ there is $\pi \in \Pi$ such that $a, \bar{c}\ \pi\ b, \bar{c}$.

The mappings for the Galois connection can now be defined:

- $\mathrm{Sim}(q)$ is the set of similarities π such that all relations and quantifiers in q are $\sim_q$-invariant under π.
- $\mathrm{Inv}(\Pi)$ is the set of all relations R and quantifiers Q on Ω which are $\approx_\Pi$-invariant under all similarities in Π.

First half of the correspondence

Let the blow-up $\hat{Q}$ of Q relative to $\sim$ be $\{\hat{R} \mid R \in Q\}$, where

$$\hat{R} = \{\, \langle a_1, \ldots, a_k\rangle \mid \exists\langle b_1, \ldots, b_k\rangle \in R, a_1 \sim b_1, \ldots a_k \sim b_k \,\}.$$

Theorem
Let q be a set of operators then

1. $Q \in \mathrm{Inv}(\mathrm{Sim}(q))$ iff $\hat{Q}$ is definable in $\mathscr{L}^{-}_{\infty\infty}(q)$.
2. $R \in \mathrm{Inv}(\mathrm{Sim}(q))$ iff R is definable in $\mathscr{L}^{-}_{\infty\infty}(q)$.

More definitions

- Π is a monoid with involution if it is closed under composition and taking converses.
- Π is full if it includes $\approx_\Pi$, is a monoid with involution, and closed under taking subsimilarities, i.e., such that if $\pi \in \Pi$ and $\pi' \subseteq \pi$ is a similarity then $\pi' \in \Pi$.

Lemma

- $\sim_q = \approx_{\mathrm{Sim}(q)}$ and
- If Π is full then $\sim_{\mathrm{Inv}(\Pi)} = \approx_\Pi$.

Theorem
Let Π be a set of similarity relations, then
$\mathrm{Sim}(\mathrm{Inv}(\Pi))$ is the smallest full monoid including Π.

Summary

$\{\, G \subseteq \mathrm{Sym}(\Omega) \,\}$ $\rightleftharpoons$ $\{\, q$ set of quantifiers on $\Omega \,\}$
least group — definability in $\mathscr{L}_{\infty\infty}$

$\{\, \Pi$ set of similarities on $\Omega \,\}$ $\rightleftharpoons$ $\{\, q$ set of quantifiers on $\Omega \,\}$
least full monoid — $\hat{\ }$-definability in $\mathscr{L}^{-}_{\infty\infty}$

Thank you for your attention.

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