

FOUNDATIONAL ASPECTS ON TEAM SEMANTICS

Nordic Congress of Mathematicians, Espoo

Fredrik Engström

August 21, 2022

University of Gothenburg

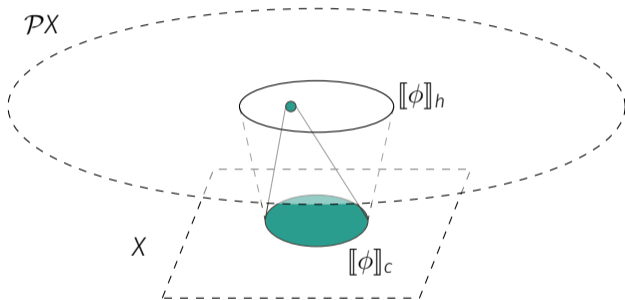
Intro

- Henkin -61: Branching, skolemization
- Hintikka -73: Part of natural languages
- Barwise -79: Branching of generalized quantifiers
- Hintikka/Sandu -89: IF-logic, game theoretic semantics
- Hodges -97: Compositional semantics for IF-logic
- Väänänen -07: Dependence Logic
- Do not first-order logic within a first-order syntax.
- By using evaluating formulas w.r.t. sets of valuations instead of single valuations.
- Can directly express independence, dependence, information flow, etc.

Investigate the construction to get a better understanding of it and the logical constants.

Lifting Tarski, as done by Hodges

- $s \models \phi$
- X is the set of all valuations (or assignments or worlds).
- $\llbracket \phi \rrbracket_c = \{ s \in X \mid s \models \phi \}$
- $\llbracket \phi \rrbracket_c \in \mathcal{P}X$
- $\llbracket \phi \rrbracket_h \in \mathcal{P}\mathcal{P}X$
- **Flatness principle:** $\llbracket \phi \rrbracket_h = \mathcal{P}\llbracket \phi \rrbracket_c$
- Hodges lift: $\lambda_h : \mathcal{P}X \rightarrow \mathcal{P}\mathcal{P}X$,
 $a \mapsto \mathcal{P}a$



A **team** is a set of valuations (or assignments or worlds).

The construction, abstractly

In Dependence Logic denotations are **closed downwards**. Replace $\mathcal{P}\mathcal{P}X$ with $\mathcal{L}\mathcal{P}X$, the set of downward-closed subsets of $\mathcal{P}X$:

$$\lambda_h : \mathcal{P}X \rightarrow \mathcal{L}\mathcal{P}X$$

- $(\mathcal{P}X, \subseteq, \cup)$ is a partially ordered monoid (POM). Let M be any POM.
- LM POM on $\mathcal{L}M$ where $A \cdot B = \{ a \cdot b \mid a \in A, b \in B \}$ ordered by subset relation.

LM is a unital commutative quantale (QTL):

- POM with unit
- Complete lattice
- Monoid operation distributes over supremum

A canonical lift

Abramsky and Väänänen (2009): The Hodges construction is **canonical**

- L is a functor from POM to QTL.
- L is the left adjoint to the forgetful functor: $\text{QTL} \rightarrow \text{POM}$, thus **canonical**.
- The unit of the adjunction is λ_h .
- Applying L and λ_h to Tarskian semantics ($\mathcal{P}\mathcal{X}$) gives Hodges semantics.

Canonical semantical construction? I

Lück (2020): L and λ_h do not uniquely lift the logical operators

There are more than one operator $\bar{x}: (\mathcal{L}\mathcal{P}X)^2 \rightarrow \mathcal{L}\mathcal{P}X$ such that

$$\lambda_h(a) \bar{x} \lambda_h(b) = \lambda_h(a \star b)$$

- Proved by observing that λ_h is not onto.

Canonical semantical construction? II

- Denotations of formulas in many logics (such as Independence Logic) are **not** downward-closed.
- Replacing $\mathcal{L}M$ by $\mathcal{P}M$ in the above argument doesn't work:

$P : M \rightarrow \mathcal{P}M$ as a functor from POM to QML is not canonical.

$P : M \rightarrow \mathcal{P}M$ as a functor from **Boolean algebras** to complete BAs is not canonical.

In fact, no free complete Boolean algebra on infinitely many generators exists.

- The Hodges lift doesn't seem to be **canonical** when using $\mathcal{P}M$.
- Are there other ways of lifting semantics?

Alternative: Singleton lift, motivation

Hintikka and Barwise:

Need branching of generalized quantifiers for doing formal semantics of natural languages.

- Adding **monotone increasing** generalized quantifiers to the Hodges construction can be done conservatively.
- However, the Hodges construction implies monotonicity of the quantifiers:

Theorem

A non-monotone increasing generalized quantifier gets the same truth condition in Dependence Logic as its monotone closure.

Example: $\exists^{=5}$ gets the same truth condition in DL as $\exists^{\geq 5}$.

Alternative: Singleton lift

- Alternative lift: $\lambda_s : \mathcal{P}X \rightarrow \mathcal{P}\mathcal{P}X, a \mapsto \{ a \}$.

Closely related to 1-semantics of Nurmi (2009).

Theorem

First-order logic with team semantics using the singleton lift extended with some dependence atom has the same strength as Dependence logic (and Σ_1^1) and can be extended with any generalized quantifier in a conservative way.

The Boolean algebra of teams

\mathcal{PPX} is a Boolean algebra.

- Boolean operators: $\perp, \neg, \vee, \wedge$
- **Internal** Boolean operators: $\bar{\perp}, \bar{\neg}, \bar{\vee}, \bar{\wedge}$:
 - $\bar{\perp} = \{ \emptyset \}$ and $\bar{\neg}A = \{ a^c \mid a \in A \}$
 - $A \bar{\vee} B = \{ a \cup b \mid a \in A, b \in B \}$
 - $A \bar{\wedge} B = \{ a \cap b \mid a \in A, b \in B \}$
- To get a propositional logic we need denotations for atoms: p_i .
- A lift $\lambda : \mathcal{PX} \rightarrow \mathcal{PPX}$ does that:

- $\llbracket \phi \rrbracket_\lambda = h_\lambda(\phi)$, where h_λ is the homomorphism from the free algebra of formulas to \mathcal{PPX} given by $h_\lambda(p_i) = \lambda(\llbracket p_i \rrbracket_c)$.
- $\phi \models_\lambda \psi$ iff $h_\lambda(\phi) \subseteq h_\lambda(\psi)$.

Substitutionality

$$\phi \vDash \psi \Rightarrow \phi[\sigma/p] \vDash \psi[\sigma/p]$$

Basic idea of logic, at least, since Bolzano:

A proposition is **universally valid** if all **variants** are true.

- Substitutionality helps to define nice proof systems.
- Substitutionality needed to apply standard techniques from **algebraic logic**.

A lift **limits the possible denotations** of atoms: the logics won't (in general) be substitutional.

The propositional logic of teams

- Quantify over **all** lifts: $\phi \models \psi$ iff $\phi \models_\lambda \psi$ for all λ .
- $\phi \models \psi$ iff $h(\phi) \subseteq h(\psi)$ for all homomorphisms h from the free algebra of formulas to \mathcal{PPX} .

Theorem (Lorimer Olsson, 2022)

The *propositional logic of teams*

- *is substitutional,*
- *has the same logical strength as Propositional Dependence logic, and*
- *there are axioms FAA s.t. $\Gamma \models_{\lambda_h} \phi$ iff $\Gamma, \text{FAA} \models \phi$.*

(FAA is not closed under substitution.)

Further work

- Find a nice proof system for the propositional logic of teams using labelled sequents.
- Apply the techniques from abstract algebraic logic to find the correct algebraization of the logic.
- Can something similar be done for first-order logic?

THANKS!

- Samson Abramsky and Jouko Väänänen. From if to bi. *Synthese*, 167(2):207–230, 2009.
- Fredrik Engström. Generalized quantifiers in dependence logic. *Journal of Logic, Language and Information*, 21:299–324, 2012. ISSN 0925-8531.
- Wilfrid Hodges. Compositional semantics for a language of imperfect information. *Logic Journal of the IGPL*, 5(4):539–563, 1997.
- Orvar Lorimer Olsson. Monadic semantics, team logics and substitution. Master's thesis, University of Gothenburg, 2022. URL <https://hdl.handle.net/2077/72005>.
- Martin Lück. *Team logic: axioms, expressiveness, complexity*. PhD thesis, Hannover: Institutionelles Repositorium der Leibniz Universität Hannover, 2020.
- Ville Nurmi. *Dependence logic: Investigations into higher-order semantics defined on teams*. PhD thesis, 2009.
- Jouko Väänänen. *Dependence logic: A new approach to independence friendly logic*, volume 70. Cambridge University Press, 2007.