# Foundational principles of team semantics

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# Outline

- Introduction to dependence logic.
- What is this construction and why this definition?
- Give an alternative approach.
- What are the foundational implications of this approach?
- Constancy: Another potential approach to single out one semantics.
- Conclusions and questions.

# Dependence Logic

$$\forall x \exists y \forall z \exists w\, Rxyzw$$

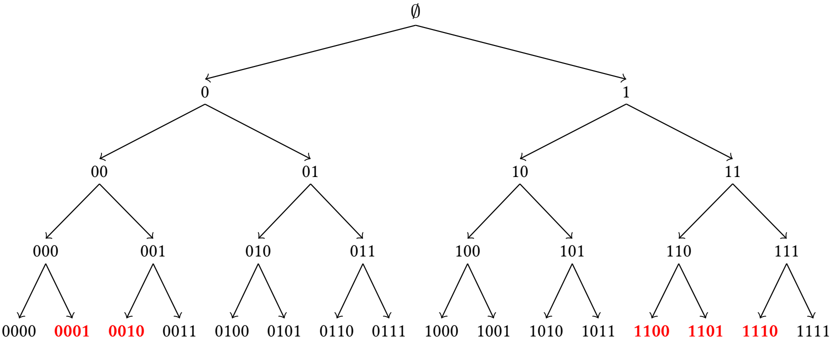
$$\begin{pmatrix} \forall x \exists y \\ \forall z \exists w \end{pmatrix} Rxyzw$$

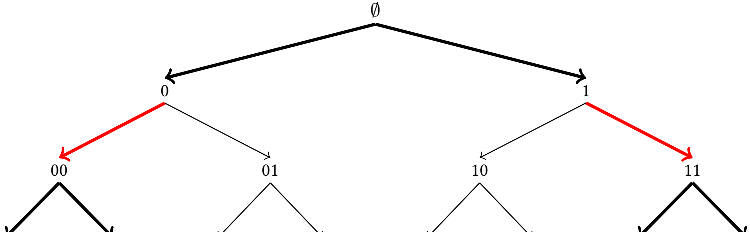
$$\forall x \exists y \forall z \exists w / x\, Rxyzw$$

$$\forall x \exists y \forall z \exists w\ (D(z, w) \land Rxyzw)$$

$\forall x \exists y \forall z \exists w Rxyzw$

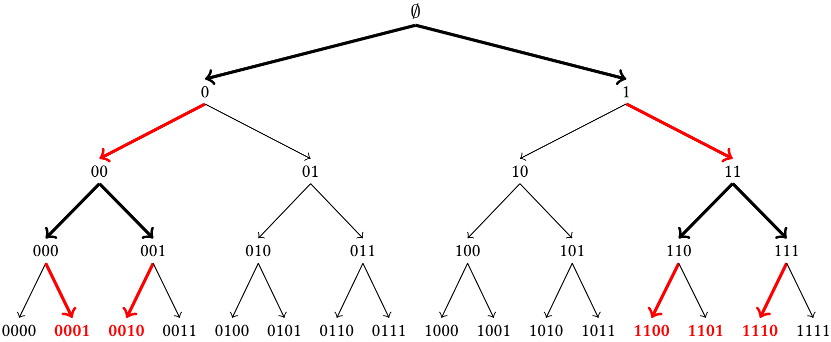
Domain is $\{ 0, 1 \}$.

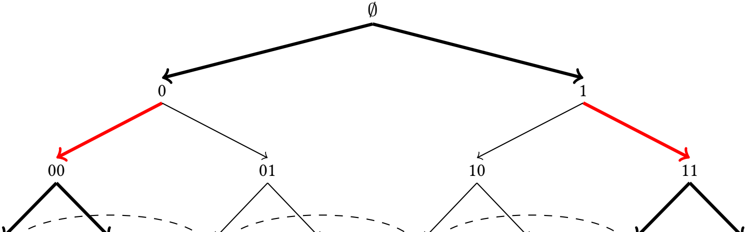


# Domain $\{ 0, 1 \}$. $\begin{pmatrix} \forall x \exists y \\ \forall z \exists w \end{pmatrix} Rxyzw$

Domain is $\{ 0, 1 \}$.



| $x$ | $y$ | $z$ | $w$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 |

$\not\models D(z, w)$

| $x$ | $y$ | $z$ | $w$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 0 |

$\models D(z, w)$

$$\begin{pmatrix} \forall x \exists y \\ \forall z \exists w \end{pmatrix} Rxyzw \equiv \forall x \exists y \forall z \exists w \, (D(z, w) \wedge Rxyzw)$$

**Definition**

$X$ a **team** = set of assignments.

# Team semantics

- **Team semantics:** Denotations of formulas are **sets of sets of assignments** (or sets of teams).
- **Flatness of FO:** A first-order formula is satisfied by a team iff all assignments satisfy the formula. (conservativeness)
- **Downward closure:** If a team satisfies a formula so does each subteam.

The downward closure property fails in some dependence like logics, e.g., independence logic and exclusion logic.

# Dependence Logic

$$\phi ::= \text{At} \mid \neg\text{At} \mid D(\bar{x}) \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x \phi \mid \forall x \phi$$

- $\mathbb{M}, X \vDash \gamma$ if for all $s \in X$: $\mathbb{M}, s \vDash \gamma$, where $\gamma$ is a literal.
- $\mathbb{M}, X \vDash D(\bar{t}, t')$ if for all $s, s' \in X$ if $s(\bar{t}) = s'(\bar{t})$ then $s(t') = s'(t')$.
- $\mathbb{M}, X \vDash \phi \wedge \psi$ if $\mathbb{M}, X \vDash \phi$ and $\mathbb{M}, X \vDash \psi$.
- $\mathbb{M}, X \vDash \phi \vee \psi$ if $\exists Y \cup Z = X$ such that $\mathbb{M}, Y \vDash \phi$ and $M, Z \vDash \psi$.
- $\mathbb{M}, X \vDash Qx\phi$ if there is $F : X \to Q_M$ s.t. $\mathbb{M}, X[F/x] \vDash \phi$.

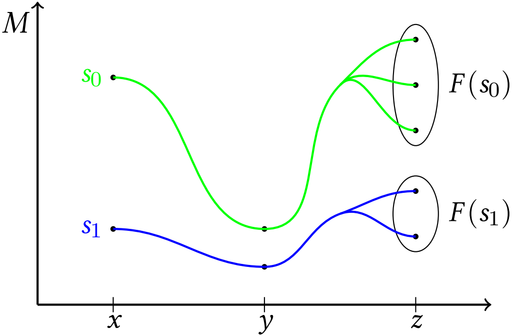
- $\forall_M = \{ M \}$
- $\exists_M = \{ A \subseteq M \mid A \neq \emptyset \}$
- $X[F/x] = \{ s[a/x] \mid s \in X \text{ and } a \in F(s) \}$
- $Q$ can be any monotone increasing type $\langle 1 \rangle$ generalized quantifier, such as $Q_0$ or $Q_\alpha$.
- $\mathbb{M} \vDash \sigma$ iff $\mathbb{M}, \{\emptyset\} \vDash \sigma$.

# Quantifiers in dependence logic

- $\mathbb{M}, X \models Qx\,\phi$ iff there is $F : X \to Q_M$ such that $\mathbb{M}, X[F/x] \models \phi$.

$$X[F/x] = \{\ s[a/x] \mid s \in X, a \in F(s)\ \}$$

**Example:** $\mathbb{M}, \{s_0, s_1\} \models \exists^{\geq 2} z\, Rxyz$



# Some properties

- $\mathbb{M}, \emptyset \vDash \phi$
- **Downwards closure:** If $Y \subseteq X$ and $\mathbb{M}, X \vDash \phi$ then $\mathbb{M}, Y \vDash \phi$.
- **Flatness:** For FO-formulas: $M, X \vDash \phi$ iff $M, s \vDash \phi$ for all $s \in X$.
- Dependence logic and Existential Second Order logic (ESO/$\Sigma^1_1$) are of the same strength.

# Foundational thoughts

# Why this semantics?

## Possible answers in general

- It is a canonical (or natural) construction (or a special case of a general construction). **No?**
- It does the intended work. **Yes?**
- It can be axiomatized. **Partially**
- It is conservative over FO. **Yes**
- It preserves logicality. **Not sure**

Is this true for the semantics of dependence logic?
What about other logics/semantics?

# What is this construction?

## The Hodges construction

- A **lift** from semantic values/denotations as sets of assignments $\mathcal{P}(X)$ to sets of sets of assignments $\mathcal{P}(\mathcal{P}(X))$.
- **Flatness**: Denotations for FO-formulas $\phi$ are $\mathcal{P}(|\phi|)$, where $|\phi|$ is the Tarskian denotation.
- **Downwards closure**: Denotations are downwards-closed sets.

Thus, lifts denotations from $\mathcal{P}(X)$ to $\mathcal{L}(\mathcal{P}(X))$ by the $\mathcal{P}$-operator.

- Abramsky and Väänänen (2009) explains the Hodges construction in terms of a monoid based semantics for linear formulas (Mitchell and Simmons, 2001).
- Lück (2020) investigates ”teamification” as a general way of lifting semantics.

# Monoid based semantics for linear logic

Mitchell and Simmons (2001):

- Commutative **ordered monoid** $M$ (POM)
- (Unital) commutative **quantale**: POM which is complete lattice and the monoid operation distributes over sup (QTL)
- The **free QTL** generated by $M$, $\mathcal{L}M$, is the set of downwards-closed sets of $M$:
  - $\downarrow X = \{ y \in M \mid y \leq x \}$
  - $X \otimes Y = \downarrow \{ x + y \mid x \in X, y \in Y \}$
  - Unit is $\{ 0 \}$.
  - $X \leq Y$ iff $X \subseteq Y$ (and $\bigvee = \bigcup$)
- This construction is a functor: Pom $\rightarrow$ Qtl.
- $\mathcal{L}$ is the **left adjoint** of the forgetful functor: Qtl $\rightarrow$ Pom.

Any commutative quantale interprets intuitionistic linear logic with, for example:

$$|\phi \otimes \psi| = |\phi| \otimes |\psi|$$

# The Hodges construction

Hodges semantics is the $\mathcal{L}$-image of Tarskian semantics.
Abramsky and Väänänen (2009) argues the Hodges construction is **canonical**, or "forced":

- Hodges semantics is the image under $\mathcal{L}$ of Tarskian semantics. "Thus these definitions are **forced**."
- Quantifiers are adjoints to substitution/projection in the Hodges semantics.
- $\mathcal{L}$ is the left adjoint to the forgetful functor: $\mathrm{POM} \to \mathrm{QNL}$ so canonical.
- The monoid based semantics for linear logic is similar.

## Abstract team logic

Lück (2020) defines **abstract team logic** as a general algebraic semantics on $\mathcal{P}(X)$.

- A team semantics is the **teamification** of a Tarskian semantics if it commutes with $\mathcal{P}$, i.e., is it's based on the flatness principle.
- There are several teamifications of Tarskian semantics.
- Thus, the Hodges semantics is not "forced" even if we submit to the flatness principle.

An operator $O : \mathcal{P}\mathcal{P}X \to \mathcal{P}\mathcal{P}X$ is a teamification of $o : \mathcal{P}X \to \mathcal{P}X$ if

$$\mathcal{P}(o(A)) = O(\mathcal{P}(A)).$$

# Problems

This construction is based on $\mathcal{LP}X$ being the right space for denotations and the flatness principle as the basic guiding principle.

## Some problems

- The Hodges construction does not seem to be canonical or "forced"
- Denotations of independence atoms, exclusion atoms and others are **not** downwards-closed.
- Generalized quantifiers can be lifted to Hodges semantics but only for monotone increasing generalized quantifiers.

# Non-monotone quantifiers

$$\mathbb{M} \models \exists^{=5} x\, Px$$

$$\exists F : \{\, \emptyset \,\} \to \exists_M^{=5}, \text{ s.t. } \mathbb{M}, \{\, \emptyset \,\}\, [F/x] \models Px$$

$$\exists A \subseteq M, \text{ s.t. } |A| = 5 \text{ and } A \subseteq P^M$$

$$\mathbb{M} \models \exists^{\geq 5} x\, Px$$

$\phi$ is satisfied by $X$ if

- every assignment $s \in X$ satisfies $\phi$.
- for every assignment $s : \mathrm{dom}(X) \to M$, $s \in X$ **iff** $s$ satisfies $\phi$.

Idea: Get rid of the flatness principle and replace $\mathcal{LP}X$ with $\mathcal{PP}X$.

# Alternative approach

# Dependence logic, take II

$$\phi ::= \mathrm{At} \mid \neg\mathrm{At} \mid D(\bar{x}) \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x\, \phi \mid \forall x\, \phi$$

- $\mathbb{M}, X \vDash \gamma$ if for all $s \in X$: $\mathbb{M}, s \vDash \gamma$, where $\gamma$ is a literal.
- $\mathbb{M}, X \vDash \phi \wedge \psi$ iff $\exists Y, Z$ s.t. $X = Y \cap Z$; $\mathbb{M}, Y \vDash \phi$ and $\mathbb{M}, Z \vDash \psi$
- $\mathbb{M}, X \vDash \phi \vee \psi$ iff $\exists Y, Z$ s.t. $X = Y \cup Z$; $\mathbb{M}, Y \vDash \phi$ and $\mathbb{M}, Z \vDash \psi$
- $\mathbb{M}, X \vDash \exists x \phi$ iff $\exists Y$ s.t. $x \in \mathrm{dom}(Y), \exists x Y = X$ and $\mathbb{M}, Y \vDash \phi$
- $\mathbb{M}, X \vDash \forall x \phi$ iff $\exists Y$ s.t. $x \in \mathrm{dom}(Y), \forall x Y = X$ and $\mathbb{M}, Y \vDash \phi$

$$Qx\, X = \{\, s : \mathrm{dom}(X) \setminus \{\, x \,\} \to M \mid \{\, a \in M \mid s[a/x] \in X \,\} \in Q_M \,\}$$

- $\mathbb{M}, X \vDash D(\bar{t}, t')$ if for all $s, s' \in X$ if $s(\bar{t}) = s'(\bar{t})$ then $s(t') = s'(t')$.

# Team logic / 1-semantics (simplified)

Similar to 1-semantics in Nurmi et al. (2009).

$$\phi ::= \mathrm{At} \mid \neg\mathrm{At} \mid \top(\bar{x}) \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x \phi \mid \forall x \phi$$

- **$\mathbb{M}, X \vDash \gamma$ if for all $s$: $s \in X$ iff $\mathbb{M}, s \vDash \gamma$, where $\gamma$ is a literal.**
- $\mathbb{M}, X \vDash \phi \wedge \psi$ iff $\exists Y, Z$ s.t. $X = Y \cap Z$; $\mathbb{M}, Y \vDash \phi$ and $\mathbb{M}, Z \vDash \psi$
- $\mathbb{M}, X \vDash \phi \vee \psi$ iff $\exists Y, Z$ s.t. $X = Y \cup Z$; $\mathbb{M}, Y \vDash \phi$ and $\mathbb{M}, Z \vDash \psi$
- $\mathbb{M}, X \vDash \exists x\phi$ iff $\exists Y$ s.t. $x \in \mathrm{dom}(Y), \exists xY = X$ and $\mathbb{M}, Y \vDash \phi$
- $\mathbb{M}, X \vDash \forall x\phi$ iff $\exists Y$ s.t. $x \in \mathrm{dom}(Y), \forall xY = X$ and $\mathbb{M}, Y \vDash \phi$

- $\mathbb{M}, X \vDash \top(\bar{x})$ iff $\exists \bar{x} X = \{ \emptyset \} [M^k / \mathrm{dom}(X) \setminus \{ \bar{x} \}]$

# Properties

**Foundational principle**

A FO-formula $\phi$ is satisfied by a team $X$ if for every assignment $s : \mathrm{dom}(X) \to M^k$, $s \in X$ iff $s$ satisfies $\phi$.

**Expressive power**

The expressive power of team logic is that of existential second-order logic.

**Generalized quantifiers**

We can add non-monotone increasing generalized quantifiers.

# Revisit the foundations

# Revisit the basic principles

- Now the lift is $\mathcal{I}(X) = \{ X \}$ instead of $\mathcal{P}$.
- Can this construction be generalized to POMs:
  - Given a POM $M$ the $\mathcal{I}(M)$ is structure on $\mathcal{P}(M)$:
  - $\mathcal{I} : M \to \mathcal{I}(M)$.
  - $X \otimes Y = \{ x + y \mid x \in X, y \in Y \}$
  - $X \leq Y$ iff $\forall x \in X \exists y \in Y (x \leq y)$
- Does $\mathcal{I}(M)$ has enough structure for defining a semantics?
- Is this construction a functor: POM $\to$ XXX? With which properties?

Is this construction generalizable to a larger class of logics?

# Logical constants

# Logicality

**When is an operator logical?**

- Topic neutrality - invariance
- Inferentialism - harmony
- Combined approaches (Feferman, Westerståhl/Bonnay)

- Are the logicality criteria preserved when lifting to team semantics?
- Are there more (too many?) logical constants in team semantics?

# Invariance

### Tarski's thesis

A quantifier, or operator, on a domain $M$ is a logical constant iff it is invariant under all **permutations** of $M$.

### Mostowski's thesis

A quantifier $Q$ is a logical constant iff it is invariant under all **bijections** (across domains).

### Theorem (McGee -91 / Krasner -38)

$Q$ is bijection invariant iff for each $\kappa$ there is a formula in $\mathcal{L}_{\infty\infty}$ defining $Q_\kappa$.

# Invariance II

**Definition**

A quantifier Q is invariant under **preimages of surjections** if for every $h : M \to N$ surjection and for all $R \subseteq N^k$: $h^{-1}(R) \in Q_M$ iff $R \in Q_N$.

**Theorem (Feferman)**

Quantifiers of type $\langle 1, \dots, 1 \rangle$ are invariant under **preimages of surjections** iff they are definable in $\mathcal{L}^{-}_{\omega\omega}$.

**Feferman's thesis -99**

A quantifier is a logical constant iff it can de defined (in typed $\lambda$-calculus) from equality and monadic quantifiers invariant under preimages of surjections.

- Are these (and similar) properties preserved under the lifts?

# Recap and open Qs

- Some arguments for Hodges semantics:
  - The Hodges lift is the left adjoint of the forgetful functor (and thus a free construction) QML $\to$ POM.
  - The semantics is the image of the Tarskian semantics.
  - The quantifiers are adjoints of substitution.
  - (It works/has nice properties.)
- Do these conditions uniquely single out Hodges semantics?
- Are there other ways to single out a team semantics? (Logicality?)
- What are the connections with other logical frameworks (with algebraic semantics)?

# That's all folks!

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