Dependence logic and generalized quantifiers

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$$\forall x \exists y \forall z \exists w Rxyzw$$

$$\downarrow$$

$$\forall z$$

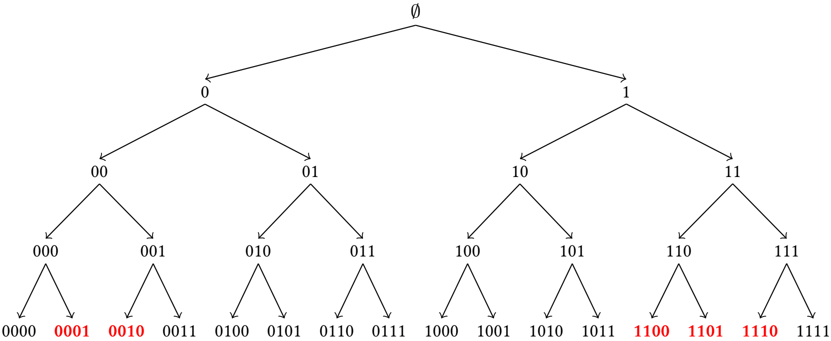
$$\downarrow$$

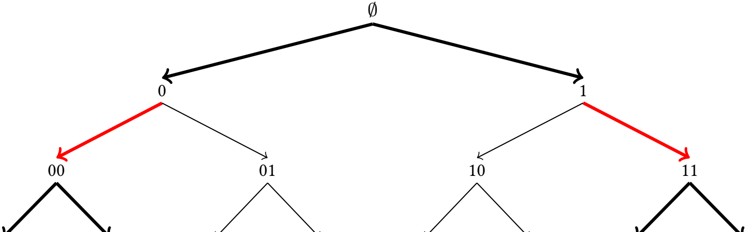
$$\exists w$$

$$\begin{pmatrix} \forall x \exists y \\ \forall z \exists w \end{pmatrix} Rxyzw$$

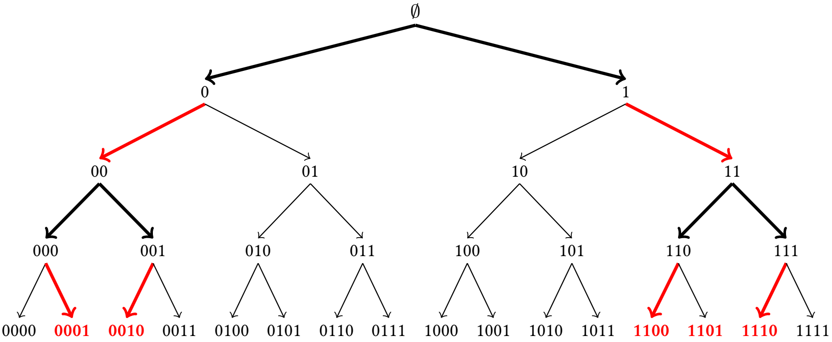
$$\begin{array}{cc} \forall z & \forall x \\ \downarrow & \downarrow \\ \exists w & \exists y \end{array}$$

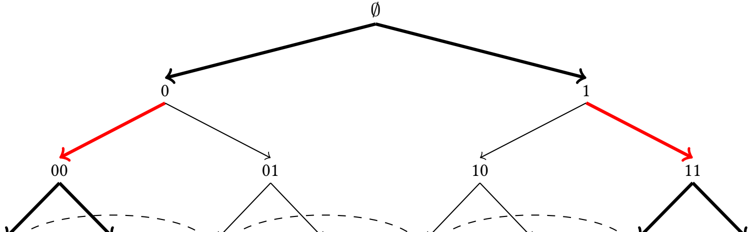
Domain $\{ 0, 1 \}$. $\forall x \exists y \forall z \exists w Rxyzw$





Domain $\{ 0, 1 \}$. $\begin{pmatrix} \forall x \exists y \\ \forall z \exists w \end{pmatrix} Rxyzw$





x	y	z	w
0	0	0	1
0	0	1	0
1	1	0	0
1	1	1	0

$\not\models D(z, w)$

x	y	z	w
0	0	0	1
0	0	1	0
1	1	0	1
1	1	1	0

$\models D(z, w)$

$$\begin{pmatrix} \forall x \exists y \\ \forall z \exists w \end{pmatrix} Rxyzw \equiv \forall x \exists y \forall z \exists w \,(D(z, w) \wedge Rxyzw)$$

Definition

X a **team** = set of assignments.

Team semantics

- **Team semantics:** Lifting semantic values (of formulas) from sets of assignment to **sets of sets of assigments** (sets of teams).
- **Flatness property of FO:** A first-order formula is satisfied by a team iff all assignments satisfy the formula.
- **Subteam property:** If a team satisfies a formula so does each subteam.

The subteam property fails in some logics, e.g., independence logic and exclusion logic.

Dependence Logic

$$\phi ::= \mathrm{At} \mid \neg\mathrm{At} \mid D(\bar{x}) \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x \phi \mid \forall x \phi$$

- $\mathbb{M}, X \vDash \gamma$ if for all $s \in X$: $\mathbb{M}, s \vDash \gamma$, where γ is a literal.
- $\mathbb{M}, X \vDash D(\bar{t}, t')$ if for all $s, s' \in X$ if $s(\bar{t}) = s'(\bar{t})$ then $s(t') = s'(t')$.
- $\mathbb{M}, X \vDash \phi \wedge \psi$ if $\mathbb{M}, X \vDash \phi$ and $\mathbb{M}, X \vDash \psi$.
- $\mathbb{M}, X \vDash \phi \vee \psi$ if there are $Y \cup Z = X$ such that $\mathbb{M}, Y \vDash \phi$ and $M, Z \vDash \psi$.
- $\mathbb{M}, X \vDash Qx\phi$ if there is $F : X \to Q_M$ s.t. $\mathbb{M}, X[F/x] \vDash \phi$.

- $\forall_M = \{ M \}$
- $\exists_M = \{ A \subseteq M \mid A \neq \emptyset \}$
- $X[F/x] = \{ s[a/x] \mid s \in X \text{ and } a \in F(s) \}$
- $\mathbb{M} \vDash \sigma$ iff $\mathbb{M}, \{\emptyset\} \vDash \sigma$.

Some properties

- $\mathbb{M}, \emptyset \vDash \phi$
- **Downwards closure:** If $Y \subseteq X$ and $\mathbb{M}, X \vDash \phi$ then $\mathbb{M}, Y \vDash \phi$.
- **Local / Context independent:** $\mathbb{M}, X \vDash \phi$ iff $\mathbb{M}, Y \vDash \phi$ where $Y = \{ s \upharpoonright \mathrm{fv}(\phi) \mid s \in X \}$.
- Dependence logic (and IF-logic) and Existential Second Order logic (ESO) is of the same strength.
- For formulas the situation is slightly different: Dependence logic and the negative fragment of ESO are equivalent.
- **Extra feature of D:** Truth is definable.

Generalized quantifiers

A generalized quantifier Q is a class of structures closed under isomorphisms.

- $Q_M = \{ R \mid (M, R) \in Q \}$.

$$Q_M \subseteq \mathcal{P}(M).$$

$$\mathbb{M}, s \models Qx\,\phi \text{ iff } \phi^{\mathbb{M},s} \in Q_M$$

- $\forall_M = \{ M \}$
- $\exists_M = \{ A \subseteq M \mid A \neq \emptyset \}$
- $(Q_1)_M = \{ A \subseteq M \mid |A| \geq \aleph_1 \}$

Q is **monotone increasing** if $A \subseteq B$ and $A \in Q_M$ implies $B \in Q_M$.

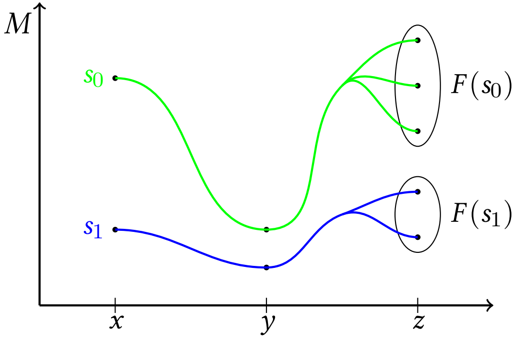
Generalized quantifiers in dependence logic

Works well only for **monotone increasing generalized quantifiers**.

- $\mathbb{M}, X \vDash Qx\,\phi$ iff there is $F : X \to Q_M$ such that $\mathbb{M}, X[F/x] \vDash \phi$.

$$X[F/x] = \{\, s[a/x] \mid s \in X, a \in F(s) \,\}$$

Example: $\mathbb{M}, \{s_0, s_1\} \vDash \exists^{\geq 2} z\, Rxyz$



Iteration and branching

Iteration
$(Q_1 \cdot Q_2)_M = \{ R \subseteq M^2 \mid \{ a \mid {}_aR \in (Q_2)_M \} \in (Q_1)_M \}$

$$(Q_1 \cdot Q_2)xy\phi \equiv Q_1 x\, Q_2 y\, \phi$$

For monotone increasing quantifiers:

$\mathrm{Br}(Q_1, Q_2)_M = \{ R \subseteq M^2 \mid A \times B \subseteq R, A \in (Q_1)_M, B \in (Q_2)_M \}$

$$\mathrm{Br}(Q_1, Q_2)xy\phi \equiv \begin{pmatrix} Q_1 x \\ Q_2 y \end{pmatrix} \phi$$

Properties of $D(Q)$

- **Empty set property:** $\mathbb{M}, \emptyset \models \phi$
- **Downwards closure:** If $Y \subseteq X$ and $\mathbb{M}, X \models \phi$ then $\mathbb{M}, Y \models \phi$.

Flattness of $FO(Q)$

$$\mathbb{M}, X \models \phi \text{ iff for all } s \in X, \mathbb{M}, s \models \phi$$

for all FO(Q)-formulas ϕ.

Respect iteration

$$\mathbb{M}, X \models (Q_1 \cdot Q_2)xy\phi \text{ iff } \mathbb{M}, X \models Q_1 x Q_2 y \phi$$

Express branching

$$D(Q) \equiv D(Q, \mathrm{Br}(Q, Q))$$

Normal form

Theorem: Normal form for $D(Q, \check{Q})$

Every $D(Q, \check{Q})$ formula is equivalent to one of the form:

$$\mathcal{H}^1 x_1 \dots \mathcal{H}^m x_m \exists y_1 \dots \exists y_n \Big(\bigwedge_{1 \leq i \leq n} D(\overline{x}^i, y_i) \wedge \theta \Big),$$

where $\mathcal{H}^i$ is either Q, $\check{Q}$ or $\forall$, and θ is a quantifier-free FO formula.

Dependence logic

- Dependence relations can be axiomatized (Armstong).
- Dependence logic has the same strength as ESO.
- The relation $\Gamma \vDash \phi$ is not r.e.
- Restricting to ϕ's without dependence atoms gives an r.e. entailment relation.
- An explicit axiomatization has been given by Kontinen and Väänänen.

Idea:

- Construct a natural deduction system in which the normal form can be derived.
- Allow dependencies in normal forms to be replaced by **finite approximations**.
- Show that in enough models (recursively saturated) the set of finite approximations is equivalent to the original sentence.

Axiomatizing $D(Q, \check{Q})$ I: General rules

First: Some rules sound for **any interpretation** of Q (monotone increasing).

- Standard rules for $\mathrm{FO}(Q, \check{Q})$ formulas.
- Standard rules for conjunction, existential quantifier, and universal quantifier.
- Commutativity, associativity and monotonicity of disjunction.
- Monotonicity, extending scope, and renaming of bound variables for Q and $\check{Q}$.
- Duality of $\check{Q}$ with respect to $\mathrm{FO}(Q, \check{Q})$ formulas.

Axiomatizing $D(Q, \check{Q})$ II: Dependence related rules

- Unnesting:

$$\frac{D(t_1, ..., t_n)}{\exists z(D(t_1, ..., z, ..., t_n) \wedge z = t_i)}$$

where z is a new variable.

- Dependence distribution:

$$\frac{\exists y_1 \ldots \exists y_n(\bigwedge_{1 \leq j \leq n} D(\bar{z}^j, y_j) \wedge \phi) \vee \exists y_{n+1} \ldots \exists y_m(\bigwedge_{n+1 \leq j \leq m} D(\bar{z}^j, y_j) \wedge \psi)}{\exists y_1 \ldots \exists y_m(\bigwedge_{1 \leq j \leq m} D(\bar{z}^j, y_j) \wedge (\phi \vee \psi))}$$

where ϕ and ψ are quantifier free FO formulas.

- Dependence introduction:

$$\frac{\exists x \mathcal{H} y \phi}{\mathcal{H} y \exists x (D(\bar{z}, x) \wedge \phi)}$$

where $\bar{z}$ lists the variables in $\mathrm{fv}(\phi) - \{x, y\}$ and $\mathcal{H} \in \{\forall, Q, \check{Q}\}$.

Approximations

Suppose σ is in normal form:

$$\mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m \exists y_1 \ldots \exists y_n \Big(\bigwedge_{1 \le i \le n} D(\bar{x}^i, y_i) \wedge \theta(\bar{x}, \bar{y}) \Big).$$

Let $A^k \sigma$ be

$$\forall \bar{x}_1 \exists \bar{y}_1 \ldots \forall \bar{x}_k \exists \bar{y}_k \Big(\bigwedge_{1 \le j \le k} R(\bar{x}_j) \to \bigwedge_{1 \le j \le k} \theta(\bar{x}_j, \bar{y}_j) \wedge \bigwedge_{\substack{1 \le i \le n \\ 1 \le j, j' \le k}} (\bar{x}^i_j = \bar{x}^i_{j'} \to y_{i,j} = y_{i,j'}) \Big)$$

Let $B\sigma$ be

$$\mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m R(x_1, \ldots, x_m).$$

Axiomatizing $D(Q, \breve{Q})$ III: The approximation rule

$$\dfrac{\sigma \qquad \begin{matrix}[B\sigma] & & [A^k\sigma] \\ & \ddots \quad \iddots & \\ & \psi & \end{matrix}}{\psi}\ \text{(Approx)}$$

where σ is a sentence in normal form, and R does not appear in ψ nor in any uncancelled assumptions in the derivation of ψ, except for $B\sigma$ and $A^k\sigma$.

Completeness for weak semantics

Let $\Gamma \vDash_w \phi$ mean that $\Gamma \vDash \phi$ for any monotone increasing (non-trivial) interpretation of Q (and $\check{Q}$ is interpreted as the dual of the interpretation of Q).

Theorem
This system is sound and complete wrt $\Gamma \vDash_w \phi$ where ϕ is $\mathrm{FO}(Q, \check{Q})$.

Uncountably many

- FO(Q_1) is axiomatizable, where Q_1 is "there exist uncountably many ...". (Kiesler)
- Add Keisler's rules for Q_1.

Define the **Skolem translation** $S\sigma$ of σ in normal form to be:

$$\mathcal{H}^1 x_1 \dots \mathcal{H}^m x_m \theta(f_i(\overline{x}^i)/y_i).$$

- Replace the approximation rule with the following rule

$$\dfrac{\sigma \qquad \begin{array}{c}[S\sigma]\\ \vdots \\ \psi\end{array}}{\psi}\ \text{(Skolem)}$$

Theorem

This system is sound and complete wrt $\Gamma \models \phi$ where ϕ is FO($Q_1, \check{Q}_1$).

Non-monotone quantifiers

$$\mathbb{M} \models \exists^{=5} x\, Px$$

$$\exists F : \{\emptyset\} \to \exists^{=5}_{M}, \text{ s.t. } \mathbb{M}, \{\emptyset\}[F/x] \models Px$$

$$\exists A \subseteq M, \text{ s.t. } |A| = 5 \text{ and } A \subseteq P^{M}$$

$$\mathbb{M} \models \exists^{\geq 5} x\, Px$$

ϕ is satisfied by X if

- every assignment $s \in X$ satisfies ϕ.
- ~~every assignment $s \in X$ satisfies ϕ.~~
- for every assignment $s : \mathrm{dom}(X) \to M^k$, $s \in X$ **iff** s satisfies ϕ.

Dependence logic, take II

$$\phi ::= \text{At} \mid \neg\text{At} \mid D(\bar{x}) \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x \phi \mid \forall x \phi$$

- $\mathbb{M}, X \vDash \gamma$ if for all $s \in X$: $\mathbb{M}, s \vDash \gamma$, where γ is a literal.
- $\mathbb{M}, X \vDash D(\bar{t}, t')$ if for all $s, s' \in X$ if $s(\bar{t}) = s'(\bar{t})$ then $s(t') = s'(t')$.
- $\mathbb{M}, X \vDash \phi \wedge \psi$ iff $\exists Y, Z$ s.t. $X = Y \cap Z$, and both $\mathbb{M}, Y \vDash \phi$ and $\mathbb{M}, Z \vDash \psi$
- $\mathbb{M}, X \vDash \phi \vee \psi$ iff $\exists Y, Z$ s.t. $X = Y \cup Z$, and both $\mathbb{M}, Y \vDash \phi$ and $\mathbb{M}, Z \vDash \psi$
- $\mathbb{M}, X \vDash \exists x \phi$ iff $\exists Y$ s.t. $x \in \text{dom}(Y), \exists x Y = \exists x X$ and $\mathbb{M}, Y \vDash \phi$
- $\mathbb{M}, X \vDash \forall x \phi$ iff $\exists Y$ s.t. $x \in \text{dom}(Y), \forall x Y = \exists x X$ and $\mathbb{M}, Y \vDash \phi$

$$Qx\, X = \{ s : \text{dom}(X) \setminus \{ x \} \to M \mid \{ a \in M \mid s[a/x] \in X \} \in Q_M \}$$

Team logic

$$\phi ::= \mathrm{At} \mid \neg\mathrm{At} \mid \top(\bar{x}) \mid \phi \otimes \phi \mid \phi \oplus \phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x\, \phi \mid \forall x\, \phi$$

- $\mathbb{M}, X \models \psi$ iff $\forall s : \mathrm{dom}(X) \to M(s \in X$ iff $\mathbb{M}, s \models \psi)$, for first-order atomic or negated atomic formulas ψ.
- $\mathbb{M}, X \models \top(\bar{x})$ iff $\exists \bar{x} X = \{ \emptyset \} [M^k/\mathrm{dom}(X) \setminus \{ \bar{x} \}]$
- $\mathbb{M}, X \models \phi \otimes \psi$ iff $\exists Y, Z$ s.t. $X = Y \cap Z$; $\mathbb{M}, Y \models \phi$ and $\mathbb{M}, Z \models \psi$
- $\mathbb{M}, X \models \phi \oplus \psi$ iff $\exists Y, Z$ s.t. $X = Y \cup Z$; $\mathbb{M}, Y \models \phi$ and $\mathbb{M}, Z \models \psi$
- $\mathbb{M}, X \models \phi \wedge \psi$ iff $\mathbb{M}, X \models \phi$ and $\mathbb{M}, X \models \psi$
- $\mathbb{M}, X \models \phi \vee \psi$ iff $\mathbb{M}, X \models \phi$ or $\mathbb{M}, X \models \psi$
- $\mathbb{M}, X \models \exists x \phi$ iff $\exists Y$ s.t. $x \in \mathrm{dom}(Y), \exists x Y = \exists x X$ and $\mathbb{M}, Y \models \phi$
- $\mathbb{M}, X \models \forall x \phi$ iff $\exists Y$ s.t. $x \in \mathrm{dom}(Y), \forall x Y = \exists x X$ and $\mathbb{M}, Y \models \phi$

Properties

Basic principle

A formula ϕ is satisfied by a team X if for every assignment $s : \mathrm{dom}(X) \to M^k$, $s \in X$ iff s satisfies ϕ.

Non-example: $\mathrm{dom}(X) = \{ x \}$:

$$\mathbb{M}, X \vDash \exists x(x = x)$$

A formula is **untangled** if no quantifier Qx appears in the scope of another quantifier $Q'x$ and no variable is both free and bound.

For first-order untangled ϕ and teams X s.t. $\mathrm{dom}(X) \cap \mathrm{bv}(\phi) = \emptyset$:

$$\mathbb{M}, X \vDash \phi \text{ iff } X = [\![\phi]\!]^{\mathbb{M}}_{\mathrm{dom}(X)}$$

Relationship with Dependence Logic

$X \models D(\bar{x}, y)$ iff

$$X \models \exists z\big(\forall \bar{w}(\top(\bar{x}, y) \otimes \top(\bar{x}, z)) \wedge (y = z \otimes \top(\bar{x}, \bar{w}))\big),$$

where z is not in $\bar{x}, y$ and $\bar{w}$ is $\mathrm{dom}(X) \setminus \{ \bar{x}, y, z \}$.

Define $f(\bar{w}, \phi)$ on the set of dependence logic formulas $D[\tau]$:

- $f(\bar{w}, D(\bar{x}, y)) = \exists z(\forall \bar{w}'(\top(\bar{x}, y) \otimes \top(\bar{x}, z)) \wedge (y = z \otimes \top(\bar{x}, \bar{w}'))$, where $\bar{w}' = \bar{w} \setminus \{ \bar{x}, y, z \}$ and z is not in $\bar{w}$.
- $f(\bar{w}, \phi) = \phi \otimes \top(\bar{w})$ if ψ is a literal,
- $f(\bar{w}, \phi \wedge \psi) = f(\bar{w}, \phi) \otimes f(\bar{x}, \phi)$, and
  $f(\bar{w}, \phi \vee \psi) = f(\bar{w}, \phi) \oplus f(\bar{x}, \phi)$, and
- $f(\bar{w}, \exists y\phi) = \exists y f(\bar{w}, y, \phi)$ and $f(\bar{w}, \forall y\phi) = \forall y f(\bar{w}, y, \phi)$.

Let ϕ^+ be the formula $f(\mathrm{fv}(\phi), \phi)$.

For every team X and formula ϕ of $D[\tau]$ such that $\mathrm{dom}(X) = \mathrm{fv}(\phi)$:

$$\mathbb{M}, X \models_{\mathrm{DL}} \phi \text{ iff } \mathbb{M}, X \models \phi^+.$$

Expressive power

We may, in a similar fashion, give a translation g of independence logic into team logic in such a way that

$$\mathbb{M}, X \models \phi \text{ iff } \mathbb{M}, X \models g(\phi, \mathrm{dom}(X)).$$

Galliani (2012) proved that every ESO-property can be expressed by an independence formula and thus by team logic.

For every $T[\tau]$-formula ϕ there is a Σ^1_1 formula Θ in the language of $\tau \cup \{ R \}$ such that for all $\mathbb{M}$ and X: $\mathbb{M}, X \models \phi$ iff $(\mathbb{M}, \mathrm{rel}(X)) \models \Theta$.

The expressive power of team logic is that of existential second-order logic, for both formulas and sentences.

Generalized quantifiers revisited

Let Q be of type $\langle n \rangle$ then $\mathbb{M}, X \vDash Q\bar{x}\phi$ iff there is Y such that $\bar{x} \in \mathrm{dom}(Y)$, $\mathbb{M}, Y \vDash \phi$ and $\exists\bar{x}X = Q\bar{x}Y$, where

$$Q\bar{x}Y = \{ s : \mathrm{dom}(Y) \setminus \{ \bar{x} \} \to M \mid Y_s(\bar{x}) \in Q_M \} .$$

$Y_s = \{ s' : \mathrm{dom}(Y) \setminus \mathrm{dom}(s) \to M \mid s \cup s' \in Y \}$.

Conservative over $\mathrm{FO}(Q)$:

For every untangled ϕ formula of $\mathrm{FO}(Q)$ and every team X such that $\mathrm{dom}(X) \cap \mathrm{bv}(\phi) = \emptyset$:

$$M, X \vDash \phi \text{ iff } X = [\![\phi]\!]^{\mathbb{M}}_{\mathrm{dom}(X)} .$$

Respects iteration:

$$M, X \vDash (Q_1 \cdot Q_2)xy\phi \text{ iff } M, X \vDash Q_1 x\, Q_2 x\, \phi$$

Conclusion

- Extending dependence logic with **montone increasing** generalized quantifiers is a natural and stable extension.
- There is a way to introduce non-monotone quantifiers by altering the basic semantics.

Open questions

- $T(Q) \equiv T(Q, \mathrm{Br}(Q, Q))$? (whenever $\mathrm{Br}(Q, Q)$ makes sense)
- $T(Q) \equiv ESO(Q)$?

That's all folks!

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