Foundations of Team semantics

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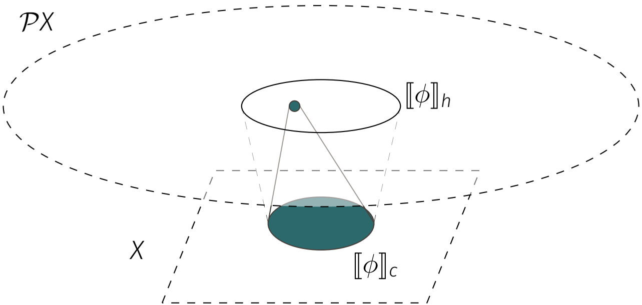
Intro

- Branching quantifiers, IF-logic, Hodges' semantics, Dependence logic
- Do non first-order logic within a first-order syntax.
- By using sets of valuations instead of single valuations.
- Can express independence, dependence, information flow.

Won't have time to present many details.

Lifting Tarski

- $s \vDash \phi$
- X is the set of all valuations/assignments/worlds.
- $[\![\phi]\!]_c = \{ s \in X \mid s \vDash \phi \}$
- $[\![\phi]\!]_c \in \mathcal{P}X$
- $[\![\phi]\!]_h \in \mathcal{P}\mathcal{P}X$
- Flatness principle: $[\![\phi]\!]_h = \mathcal{P}[\![\phi]\!]_c$
- Hodges lift: $\lambda_h : \mathcal{P}X \to \mathcal{P}\mathcal{P}X$, $a \mapsto \mathcal{P}a$



A **team** is a set of valuations/assignments/worlds.

Canonical lift

- In Dependence Logic denotations are **closed downwards**.
- Replace $\mathcal{PP}X$ with $\mathcal{LP}X$, the set of downward-closed subsets $\mathcal{P}X$:

$$\lambda_h : \mathcal{P}X \to \mathcal{LP}X$$

- M partially ordered monoid (POM), $\mathcal{L}M$ is the set of downward-closed subsets of M.
- $\mathcal{L}M$ is a commutative quantale (QTL).

Abramsky and Väänänen (2009): The Hodges construction is canonical

- $\mathcal{L}$ is a functor from POM to QTL.
- $\mathcal{L}$ is the left adjoint to the forgetful functor: QTL $\to$ POM.
- The unit of the adjunction is λ_h.
- Applying $\mathcal{L}$ and λ_h to Tarskian semantics gives you Hodges semantics.

Canonical lift?

Lück (2020): $\mathcal{L}$ and λ_h do not uniquely lift the logical operators

There are more than one operator $\bar{\star}$: $(\mathcal{LP}X)^2 \to \mathcal{LP}X$ such that

$$\lambda_h(a) \mathbin{\bar{\star}} \lambda_h(b) = \lambda_h(a \star b)$$

- Denotations of formulas in Independence logic (and other logics) are **not** downward-closed.
- The functor mapping $M \mapsto \mathcal{P}M$ isn't as nice.
- Thus, the Hodges construction doesn't seem to be **canonical** in this more general setting.
- Are there other ways of lifting semantics?

Alternative lifts

- Adding **monotone increasing** generalized quantifiers to Hodges construction can be done conservatively.
- The Hodges lift seems to imply monotonicity of the quantifiers.
- Alternative lift: $\lambda_s : \mathcal{P}X \to \mathcal{P}\mathcal{P}X, a \mapsto \{ a \}$.

Theorem

The first-order logic based on λ_s has the same strength as Dependence logic (and Σ^1_1) and can conservatively be extended with any generalized quantifier.

Boolean algebra of teams

$\mathcal{PP}X$ is a Boolean algebra.

- Boolean operators: $\bot$, $\neg$, $\vee$, $\wedge$
- Internal Boolean operators: $\bar{\bot}$, $\bar{\neg}$, $\bar{\vee}$, $\bar{\wedge}$:

$$\bar{\bot} = \{ \emptyset \}, \bar{\neg} A = \{ a^c \mid a \in A \},$$
$$A \mathbin{\bar{\vee}} B = \{ a \cup b \mid a \in A, b \in B \}, A \mathbin{\bar{\wedge}} B = \{ a \cap b \mid a \in A, b \in B \}$$

- To get a logic we need denotations for p_i.
- A lift $\lambda : \mathcal{P}X \to \mathcal{PP}X$ does that: $\phi \vDash_\lambda \psi$.

Substitutional: $\phi \vDash \psi \to \phi[\sigma/p] \vDash \psi[\sigma/p]$

A lift limits the possible denotations of atoms: the logics won't be substitutional.

$$p \vee p \vDash p, \text{ but } D(p) \vee D(p) \nvDash D(p)$$

The logic of teams

- The logic of teams: quantify over all lifts: $\phi \vDash \psi$ iff $\phi \vDash_\lambda \psi$ for all λ.
- $\phi \vDash \psi$ iff $h(\phi) \subseteq h(\psi)$ for all homomorphisms h from the free algebra of formulas to $\mathcal{PP}X$.

Theorem (Lorimer Olsson, 2022)

The logic of teams

- *is substitutional,*
- *has the same logical strength as Propositional Dependence logic, and*
- *it interprets Propositional Dependence logic.*

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