

Substitutional Logics of Teams

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The plan

- Background and motivation
 - Substitutionality
- A substitutional propositional logic using teams
 - Semantics and adequacy
 - Deductions and completeness
 - Axiomatizing standard team semantics
- A substitutional first-order logic using teams
 - Semantics
 - Axiomatizing standard team semantics

Background and motivation

Substitutionality and Dependence logic

Substitutionality:

$$\varphi \models \psi \Rightarrow h(\varphi) \models h(\psi)$$

for all homomorphisms h from the absolutely free (term) algebra of formulas to itself.

- helps when constructing uniform proof systems.
- needed to apply (standard) techniques from **algebraic logic**.

Dependence logic and its siblings are not substitutional:

$P(x) \vee P(x) \models P(x)$, but $=(x) \vee =(x) \not\models =(x)$

Due to that the possible denotations of **atoms** is a proper subset of the possible denotations of **formulas**.

The project

Basic question

Is there a substitutional logic that can express dependence and independence?

Guiding ideas:

- Start with propositional logic, for simplicity and its strong algebraic connections.
- Want the denotations of propositional variables as general as possible.
- Include “enough” connectives to be able to express standard team semantics.

A propositional logic for team properties

Boolean algebras and teams

Propositional logic is the logic of Boolean algebras

$\varphi \models \psi$ iff $h(\varphi) \leq h(\psi)$ for all homomorphisms $h : \text{Fm} \rightarrow B$ and Boolean algebras B .

What is the corresponding algebra for team semantics of propositional logic?

- Valuations are elements of $2^{\mathbb{N}}$
- Teams are elements of $\mathcal{P}(2^{\mathbb{N}}) = B$
- Denotations of formulas are elements of $\mathcal{P}(B)$.
- Logical connectives correspond to operators on $\mathcal{P}(B)$.

Algebras for team properties

- Let B be a Boolean algebra using 0 , $-$, \cdot , and $+$ as operators.
- $\mathcal{P}(B)$ is a Boolean algebra with additional point-wise operators:
 - Standard (set-theoretic) Boolean operators: \perp , \neg , \vee , \wedge
 - **Internal** (point-wise) operators: $\perp\!\!\!\perp$, \Rightarrow , \wp , $\wedge\!\!\!\wedge$
 - $\perp\!\!\!\perp = \{0\}$
 - $\Rightarrow X = \{-a \mid a \in X\}$
 - $X \wp Y = \{a + b \mid a \in X, b \in Y\}$
 - $X \wedge\!\!\!\wedge Y = \{a \cdot b \mid a \in X, b \in Y\}$
- Structures similar to $\mathcal{P}(B)$ with **point-wise operators** have been studied under different names:
 - Second-order Boolean algebras (Brink, 1984)
 - Complex algebras (Goldblatt, 1989)
 - Hyperboolean algebras (Goranko and Vakarelov, 1999)

Propositional Logic of Team Properties (LTP)

Formulas (Fm):

$$\varphi ::= \perp \mid P_i \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \perp\!\!\!\perp \mid \exists\varphi \mid \varphi \forall \varphi \mid \varphi \/\!\!/\varphi$$

Definition

$$\varphi \models \psi \text{ iff}$$

for all Boolean algebras B and all homomorphisms $H : \text{Fm} \rightarrow \mathcal{P}(B)$:

$$H(\varphi) \subseteq H(\psi).$$

$\models \psi$ iff $H(\psi) = B$ for all $H : \text{Fm} \rightarrow \mathcal{P}(B)$.

Satisfaction relation

Let B be a Boolean algebra, $v : \mathbb{N} \rightarrow \mathcal{P}B$. We define $B, v, b \models \varphi$ by the following recursive definition.

- $B, v, b \not\models \perp$,
- $B, v, b \models P_i$ if $b \in v(i)$,
- $B, v, b \models \neg\varphi$ if $B, v, b \not\models \varphi$,
- $B, v, b \models \neg\varphi$ if $B, v, -b \models \varphi$,
- $B, v, b \models \varphi \vee \psi$ if $B, v, b \models \varphi$ or $B, v, b \models \psi$,
- $B, v, b \models \varphi \wedge \psi$ if $B, v, b \models \varphi$ and $B, v, b \models \psi$,
- $B, v, b \models \varphi \vee\vee \psi$ if there are $b_1, b_2 \in B$ s.t. $b = b_1 + b_2$, $B, v, b_1 \models \varphi$ and $B, v, b_2 \models \psi$, and
- $B, v, b \models \varphi \wedge\wedge \psi$ if there are $b_1, b_2 \in B$ s.t. $b = b_1 \cdot b_2$, $B, v, b_1 \models \varphi$ and $B, v, b_2 \models \psi$.

Equivalence of the definitions

Proposition

$\varphi \vDash \psi$ iff for all Boolean algebras B , all $v : \mathbb{N} \rightarrow \mathcal{P}(B)$, and all $b \in B$, if $B, v, b \vDash \varphi$, then $B, v, b \vDash \psi$.

- When B is the Boolean algebra of sets of propositional valuations, i.e., $B = \mathcal{P}(2^{\mathbb{N}})$ and $b \in B$, then b is a team in the sense of Propositional Dependence logic.
- If also

$$v(i) = \mathcal{P} \{ s \in 2^{\mathbb{N}} \mid s(i) = 1 \}$$

then $B, v, b \vDash \varphi$ iff the team b satisfies φ in the sense of Propositional Dependence logic (formulated in a different language!)

Some results

- $P \models P \vee P$, but $P \vee P \not\models P$
- $\not\models P \vee \neg P$ and $\not\models P \vee \Rightarrow P$ [$H(P) = \{0\}$]
- LTP is substitutional. [Composition of two homomorphisms is a homomorphism.]
- LTP is conservative over propositional logic. [External connectives are defined over a Boolean algebra.]

Theorem

- The set of countable and finite Boolean algebras is adequate for LTP.
- The set of finite Boolean algebras is not adequate.
- LTP is undecidable (Knudstorp, 2025).

Labelled formulas

- Labels (**Lb**): $a ::= 0 \mid p_i \mid -a \mid a + a \mid a \cdot a$
- Labelled formulas: $a : \varphi$, where $a \in \mathbf{Lb}$, $\varphi \in \mathbf{Fm}$

Definition

$$a : \varphi \vDash b : \psi \text{ if}$$

for all B , all $h : \mathbf{Lb} \rightarrow B$ and all $H : \mathbf{Fm} \rightarrow \mathcal{P}B$: if $h(a) \in H(\varphi)$ then $h(b) \in H(\psi)$.

Observation

$$\varphi \vDash \psi \quad \text{iff} \quad p : \varphi \vDash p : \psi$$

Rules for external connectives

$$\frac{a:\varphi \quad a:\psi}{a:\varphi \wedge \psi} \wedge\text{I}$$

$$\frac{a:\varphi \wedge \psi}{a:\varphi} \wedge\text{E}$$

$$\frac{a:\varphi \wedge \psi}{a:\psi} \wedge\text{E}$$

$$\frac{a:\varphi}{a:\varphi \vee \psi} \vee\text{I}$$

$$\frac{a:\psi}{a:\varphi \vee \psi} \vee\text{I}$$

$$\frac{a:\varphi \vee \psi \quad \begin{array}{c} [a:\varphi] \\ \vdots \\ b:\sigma \end{array} \quad \begin{array}{c} [a:\psi] \\ \vdots \\ b:\sigma \end{array}}{b:\sigma} \vee\text{E}$$

$$\frac{\begin{array}{c} [a:\varphi] \\ \vdots \\ b:\perp \end{array}}{a:\neg\varphi} \neg\text{I}$$

$$\frac{a:\varphi \quad a:\neg\varphi}{a:\perp} \neg\text{E}$$

$$\frac{\begin{array}{c} [a:\neg\varphi] \\ \vdots \\ b:\perp \end{array}}{a:\varphi} \text{RAA}$$

$$\frac{a:\perp}{b:\varphi} \perp\text{E}$$

Rules for internal connectives

$a \equiv b$ is shorthand for $(-a + b) \cdot (a + -b) : \top$, where \top is $\Rightarrow \perp$

$$\frac{a : \varphi \quad b : \psi}{a \cdot b : \varphi \wedge \psi} \wedge I$$

$$\frac{a : \varphi \quad b : \psi}{a + b : \varphi \vee \psi} \vee I$$

$$\begin{array}{c} [p : \varphi] \\ [q : \psi] \\ [a \equiv p \cdot q] \\ \vdots \end{array}$$

$$\frac{a : \varphi \wedge \psi \quad b : \sigma}{b : \sigma} \wedge E$$

$$\begin{array}{c} [p : \varphi] \\ [q : \psi] \\ [a \equiv p + q] \\ \vdots \end{array}$$

$$\frac{a : \varphi \vee \psi \quad b : \sigma}{b : \sigma} \vee E$$

$$\frac{a : \varphi}{-a : \neg \varphi} \neg I$$

$$\frac{a : \neg \varphi}{-a : \varphi} \neg E$$

Rules for labels

$$\frac{a_1 : \Pi \quad \dots \quad a_k : \Pi}{b : \Pi} \text{ taut}$$

Assuming $a_1 \cdot \dots \cdot a_k \vdash b$ in propositional logic.

$$\frac{a \equiv b \quad b : \varphi}{a : \varphi} \text{ sub}$$

Soundness and completeness

$$a : \varphi \models b : \psi \quad \text{iff} \quad a : \varphi \vdash b : \psi$$

Strong propositional team logic: PT^+

Yang and Väänänen (2017)

$$\varphi ::= P_i \mid \sim P_i \mid \perp \mid \text{NB} \mid \varphi \wp \varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi$$

Valuations are elements of $2^{\mathbb{N}}$. Teams are elements of $\mathcal{P}(2^{\mathbb{N}})$.

Define $H_V : \text{Fm}_{\text{PT}^+} \rightarrow \mathcal{P}\mathcal{P}(2^{\mathbb{N}})$ by

$$H_V(P_i) = \mathcal{P} \{s \in 2^{\mathbb{N}} \mid s(i) = 1\}$$

$$H_V(\sim P_i) = \mathcal{P} \{s \in 2^{\mathbb{N}} \mid s(i) = 0\}$$

$$\text{PT}^+ : \varphi \vDash \psi \quad \text{iff} \quad H_V(\varphi) \subseteq H_V(\psi)$$

Note that $H_V(P_i)$ are **principal ideals**.

Axiomatizing PT^+ in LTP

- $\Box X = \emptyset$ if $X \neq B$; and $\Box B = B$
- $\sim X = \{b \in B \mid \forall a \in X, b \cdot a = \perp\}$

Observation

$\Box \varphi \vDash \psi$ iff for all $H : \text{Fm} \rightarrow \mathcal{P}(B)$, if $H(\varphi) = B$ then $H(\psi) = B$.

Theorem

$X \vee \sim X = B$ iff X is a principal ideal.

Principal Variable Axioms: $\text{PVA} = \{\Box(P_i \vee \sim P_i) \mid i \in \mathbb{N}\}$.

Theorem

$PT^+ : \varphi \vDash \psi$ iff LTP : PVA, $\varphi \vDash \psi$.

An attempt at a first-order logic for
team properties

A first-order logic for team properties

$$\varphi ::= \text{At} \mid x = y \mid x \neq y \mid \neg\varphi \mid \exists\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x\varphi \mid \forall x\varphi$$

Here At ranges over atomic formulas in the language of a fixed relational signature \mathcal{L} .

Models

A model in this setting consists of a non-empty domain M together with an interpretation I of the symbols in the signature \mathcal{L} such that $I(c) \in M$ for each constant symbol c , and $I(P) \subseteq \mathcal{P}(M^k)$ for each k -ary predicate symbol P .

Thus, predicate symbols are interpreted as collections of k -ary relations, rather than as single relations.

Satisfaction

- $M, X \models P(\bar{x})$ iff $X(\bar{x}) \in I(P)$
- $M, X \models x = y$ iff $\forall s \in X, s(x) = s(y)$
- $M, X \models x \neq y$ iff $\forall s \in X, s(x) \neq s(y)$
- $M, X \models \neg\varphi$ iff $M, X \not\models \varphi$
- $M, X \models \neg\varphi$ iff $M, X^c \models \varphi$
- $M, X \models \varphi \wedge \psi$ iff $\exists Y, Z$ s.t. $X = Y \cap Z; M, Y \models \varphi$ and $M, Z \models \psi$
- $M, X \models \varphi \vee \psi$ iff $\exists Y, Z$ s.t. $X = Y \cup Z; M, Y \models \varphi$ and $M, Z \models \psi$
- $M, X \models \varphi \wedge \psi$ iff $M, X \models \varphi$ and $M, X \models \psi$
- $M, X \models \varphi \vee \psi$ iff $M, X \models \varphi$ or $M, X \models \psi$
- $M, X \models \exists x\varphi$ iff $\exists F : X \rightarrow \mathcal{P}(M) \setminus \{\emptyset\}$ s.t. $M, X[F/x] \models \varphi$
- $M, X \models \forall x\varphi$ iff $M, X[M/x] \models \varphi$

Some results

- The logic is substitutional.
- Functional dependence and conditional independence are definable.
- The empty signature fragment is expressibly equivalent to full second-order logic.

Proposition

$M, X \models \Box(P(\bar{x}) \vee \sim P(\bar{x}))$ if and only if $I(P) = \mathcal{P}(R)$ for some set $R \subseteq M^k$.

Standard team semantics is axiomatized by the schema PVA:

$$\forall \bar{x} \Box(P(\bar{x}) \vee \sim P(\bar{x}))$$

for each predicate symbol P .

Embedding Dependence logic

Let φ be a formula of Dependence Logic, φ' is defined by:

- $\neg x = y \mapsto x \neq y$
- $\neg P(\bar{x}) \mapsto \sim P(\bar{x})$
- $= (\bar{x}, y) \mapsto \text{dep}(\bar{x}, y)$
- $\forall \mapsto \mathbb{W}$

Theorem

$\Gamma \models \varphi$ in Dependence logic iff $\Gamma', \text{PVA} \models \varphi'$

Further work

- What are the algebras of LTP? Is there a representation theorem? More specifically are there algebras that satisfy all LTP-equations but is not isomorphic to a substructure of a $\mathcal{P}(B)$?
- The semantical construction of LTP can be seen as “adding” a Boolean algebra on a another Boolean algebra, and the proof system is clearly propositional logic “on top” of propositional logic. Are similar constructions applicable to other logics?
- Is the first-order version expressibly equivalent to second-order logic over non-empty signatures?

THANKS!

- Chris Brink. Second-order boolean algebras. **Quaestiones Mathematicae**, 7(2):93–100, 1984.
- Fredrik Engström and Orvar Lorimer Olsson. The propositional logic of teams. **preprint arXiv:2303.14022**, 2023.
- Robert Goldblatt. Varieties of complex algebras. **Annals of pure and applied logic**, 44(3):173–242, 1989.
- Valentin Goranko and Dimiter Vakarelov. Hyperboolean algebras and hyperboolean modal logic. **Journal of Applied Non-classical Logics**, 9(2-3):345–368, 1999.
- Søren Brinck Knudstorp. Diamonds and dominoes: Impossibility results for associative modal logics. **arXiv preprint arXiv:2506.16366**, 2025.
- Davide Emilio Quadrellaro. Algebraic semantics of intuitionistic inquisitive and dependence logic. **Short Papers Advances in Modal Logic AiML 2020**, page 75, 2020.
- Fan Yang and Jouko Väänänen. Propositional team logics. **Annals of Pure and Applied Logic**, 168(7):1406–1441, 2017.