An alternative approach to Dependence logic

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Outline

- Introduction to Dependence logic.
- What is the construction and why this one?
- Give an alternative approach.
- What are the implications of this?
- Conclusions and questions.

Dependence Logic

$$\forall x \, \exists y \, \forall z \, \exists w \, Rxyzw$$

$$\begin{pmatrix} \forall x \, \exists y \\ \forall z \, \exists w \end{pmatrix} Rxyzw$$

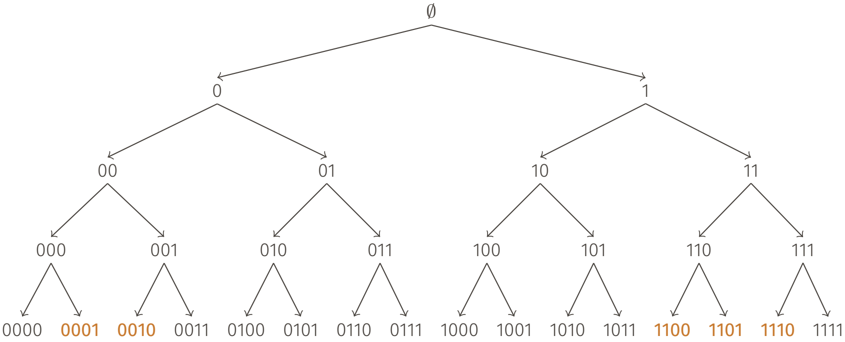
$$\exists f \, \exists g \, \forall x \, \forall z \, Rxf(x)zg(z)$$

$$\forall x \, \exists y \, \forall z \, \exists w/x \, Rxyzw$$

$$\forall x \, \exists y \, \forall z \, \exists w \, \left(D(z, w) \wedge Rxyzw\right)$$

$\forall x\, \exists y\, \forall z\, \exists w\, Rxyzw$

Domain is $\{0, 1\}$**.**

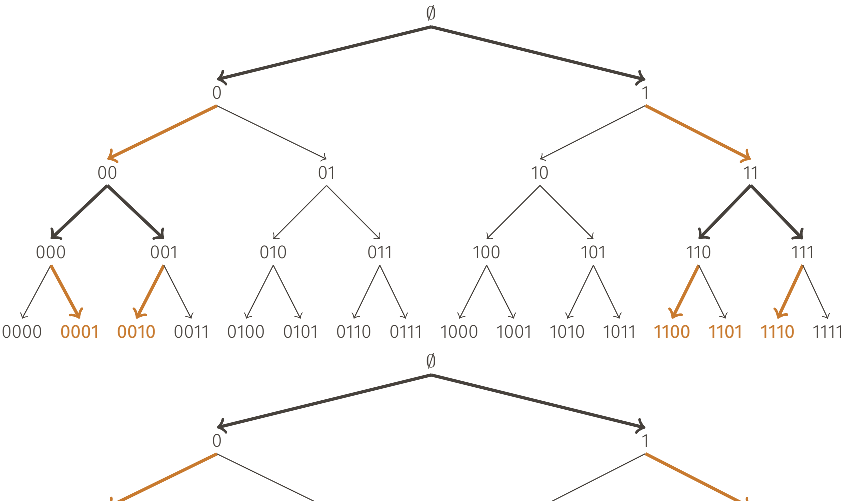


x	y	z	w
0	0	0	1
0	0	1	0
1	1	0	0

∅

$\forall x\, \exists y\, \forall z\, \exists w/x\; Rxyzw$

Domain is $\{\,0,1\,\}$.



Dependence atom

x	y	z	w
0	0	0	1
0	0	1	0
1	1	0	1
1	1	1	0

$\models D(z, w)$

Definition

X is a **team**, i.e., a set of assignments.

$\mathbb{M}, X \models D(\bar{x}, y)$ iff for all $s, s' \in X$ if $s(\bar{x}) = s'(\bar{x})$ then $s(y) = s'(y)$.

Team semantics

- A formula is satisfied by a set of assignments, a **team**.
- **Flatness:** A first-order formula is satisfied by a team iff all assignments satisfy the formula:

$X \vDash \phi \wedge \psi$ iff $\forall s \in X, s \vDash \phi \wedge \psi$ iff $\forall s \in X, s \vDash \phi$ and $s \vDash \psi$ iff $X \vDash \phi$ and $X \vDash \psi$

$X \vDash \phi \vee \psi$ iff $\forall s \in X, s \vDash \phi \vee \psi$ iff $\forall s \in X, s \vDash \phi$ or $s \vDash \psi$ iff $\exists Y \cup Z = X, Y \vDash \phi$ and $Z \vDash \psi$

Dependence Logic

$\phi ::= \text{At} \mid \neg\text{At} \mid D(\bar{x}, y) \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x\, \phi \mid \forall x\, \phi$

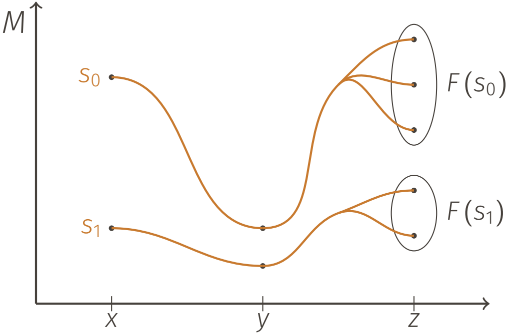
- $\mathbb{M}, X \vDash \gamma$ if for all $s \in X$: $\mathbb{M}, s \vDash \gamma$, where γ is a literal.
- $\mathbb{M}, X \vDash D(\bar{x}, y)$ if for all $s, s' \in X$ if $s(\bar{x}) = s'(\bar{x})$ then $s(y) = s'(y)$.
- $\mathbb{M}, X \vDash \phi \wedge \psi$ if $\mathbb{M}, X \vDash \phi$ and $\mathbb{M}, X \vDash \psi$.
- $\mathbb{M}, X \vDash \phi \vee \psi$ if $\exists Y \cup Z = X$ such that $\mathbb{M}, Y \vDash \phi$ and $M, Z \vDash \psi$.
- $\mathbb{M}, X \vDash Qx\, \phi$ if ..

Quantifiers in dependence logic (Engström, 2012)

- $\mathbb{M}, X \vDash Qx\,\phi$ iff there is $F : X \to Q_M$ such that $\mathbb{M}, X[F/x] \vDash \phi$.

$\forall_M = \{ M \}$, $\exists_M = \{ A \subseteq M \mid A \neq \emptyset \}$, $X[F/x] = \{ s[a/x] \mid s \in X, a \in F(s) \}$

Example: $\mathbb{M}, \{s_0, s_1\} \vDash \exists^{\geq 2} z\, Rxyz$



Some properties

- $\mathbb{M}, \emptyset \vDash \phi$
- Downwards closure: If $Y \subseteq X$ and $\mathbb{M}, X \vDash \phi$ then $\mathbb{M}, Y \vDash \phi$.
- Flatness: For FO-formulas: $\mathbb{M}, X \vDash \phi$ iff $\mathbb{M}, s \vDash \phi$ for all $s \in X$.
- Dependence logic and Existential Second Order logic (ESO/Σ_1^1) are of the same strength.

What is happening?

The Hodges construction

- A lift from denotations as sets of assignments to sets of sets of assignments: $\mathcal{P}X \to \mathcal{P}\mathcal{P}X$.
- Flatness: Denotations of FO-formulas ϕ are $\mathcal{P}[\phi]$, where $[\phi]$ is the Tarskian denotation.
- Downwards closure: Denotations are downward-closed sets: $\mathcal{P}X \to \mathcal{L}\mathcal{P}X$.

$\mathcal{L}\mathcal{P}X$ is the set of all downward-closed subsets of $\mathcal{P}X$.

Abramsky and Väänänen (2009): The Hodges construction is canonical, and forced:

- Hodges semantics is the image under $\mathcal{L}$ of Tarskian semantics, and
- $\mathcal{L}$ is the left adjoint to the forgetful functor: QTL $\to$ POM.

Problems

The construction is based on the **flatness** principle and $\mathcal{LP}X$ being the right space for denotations.

But

- Denotations of independence atoms, exclusion atoms and others are **not** downward-closed.
- The functor is not canonical if we replace $\mathcal{LP}X$ with $\mathcal{PP}X$.
- Thus, the Hodges construction is not "forced" in this more general setting.
- Generalized quantifiers can conservatively be added to Hodges semantics but only for monotone increasing generalized quantifiers.

Alternative approach – Team logic

Dependence logic, take II

$\phi ::= \text{At} \mid \neg\text{At} \mid D(\bar{x}, y) \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x\, \phi \mid \forall x\, \phi$

- $\mathbb{M}, X \vDash \gamma$ if for all $s \in X$: $\mathbb{M}, s \vDash \gamma$, where γ is a literal.
- $\mathbb{M}, X \vDash D(\bar{x}, y)$ if for all $s, s' \in X$ if $s(\bar{x}) = s'(\bar{x})$ then $s(y) = s'(y)$.
- $\mathbb{M}, X \vDash \phi \wedge \psi$ iff $\exists Y, Z$ s.t. $X = Y \cap Z$; $\mathbb{M}, Y \vDash \phi$ and $\mathbb{M}, Z \vDash \psi$
- $\mathbb{M}, X \vDash \phi \vee \psi$ iff $\exists Y, Z$ s.t. $X = Y \cup Z$; $\mathbb{M}, Y \vDash \phi$ and $\mathbb{M}, Z \vDash \psi$
- $\mathbb{M}, X \vDash Qx\, \phi$ iff $\exists Y$ s.t. $x \in \text{dom}(Y)$, $Qx\, Y = X$ and $\mathbb{M}, Y \vDash \phi$

- $Qx\, X = \{ s : \text{dom}(X) \setminus \{ x \} \to M \mid \{ a \in M \mid s[a/x] \in X \} \in Q_M \}$
- The lifting operator is $\mathcal{P}X$.

Team logic

$$\phi ::= \mathrm{At} \mid \neg\mathrm{At} \mid D(\bar{x}, y) \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x\, \phi \mid \forall x\, \phi$$

- $\mathbb{M}, X \vDash \gamma$ **if for all** s: $s \in X$ **iff** $\mathbb{M}, s \vDash \gamma$**, where** γ **is a literal.**
- $\mathbb{M}, X \vDash D(\bar{x}, y)$ if for all $s, s' \in X$ if $s(\bar{x}) = s'(\bar{x})$ then $s(y) = s'(y)$.
- $\mathbb{M}, X \vDash \phi \wedge \psi$ iff $\exists Y, Z$ s.t. $X = Y \cap Z$; $\mathbb{M}, Y \vDash \phi$ and $\mathbb{M}, Z \vDash \psi$
- $\mathbb{M}, X \vDash \phi \vee \psi$ iff $\exists Y, Z$ s.t. $X = Y \cup Z$; $\mathbb{M}, Y \vDash \phi$ and $\mathbb{M}, Z \vDash \psi$
- $\mathbb{M}, X \vDash Qx\, \phi$ iff $\exists Y$ s.t. $x \in \mathrm{dom}(Y)$, $Qx\, Y = X$ and $\mathbb{M}, Y \vDash \phi$

- $Qx\, X = \{ s : \mathrm{dom}(X) \setminus \{ x \} \to M \mid \{ a \in M \mid s[a/x] \in X \} \in Q_M \}$
- The lifting operator is $\mathcal{I}X = \{ X \}$.

Properties – Team logic

The foundational principle – replacing flatness

For FO-formulas ϕ, denotations are $\mathcal{I}[\phi] = \{\ [\phi]\ \}$.

Expressive power

The expressive power of team logic is that of existential second-order logic.

Generalized quantifiers

We can conservatively add any type of generalized quantifiers.

Substitutionality

- A lift λ of literals from $\mathcal{P}X$ to $\mathcal{P}\mathcal{P}X$ (for example $\mathcal{P}$ or $\mathcal{I}$) specifies the semantics fully.
- The lift limits the possible denotations of atoms.
- No substitutionality: $p \vee p \vDash p$, but $D(x) \vee D(x) \nvDash D(x)$
- Define **basic team logic** by quantifying over **all** lifts: $\phi \vDash \psi$ iff $\phi \vDash_\lambda \psi$ for all λ.

Theorem (Lorimer Olsson, 2022)

*The **propositional basic team logic** you get*

- *is substitutional,*
- *has the same logical strength as Propositional Dependence logic, and*
- *it interprets Propositional Dependence logic.*

Outro

Recap and open Qs

- The arguments for the Hodges lift being canonical are not fully valid.
- There are alternative lifts, such as $\mathcal{I}$.
- Quantifying over all lifts: **basic team logic**.
- The propositional basic team logic is **substitutional**.
- What is an **axiomatization** of it?
- What is an **algebraization** of it?
- Is the **first-order** basic team logic well-behaved?

That's all!

Samson Abramsky and Jouko Väänänen. From if to bi. **Synthese**, 167(2):207–230, 2009.

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