The propositional logic of teams

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Closure under substitutions

Bolzano (1837)

A proposition is universally valid if all variants are true.

$$\varphi \vDash \psi \;\Rightarrow\; h(\varphi) \vDash h(\psi)$$

h is an homomorphism from the absolutely free (term) algebra of formulas to itself.

- helps when constructing proof systems.
- needed to apply (standard) techniques from algebraic logic.

Dependence logic (and its siblings) are not closed under substitutions:

$$p \vee p \vDash p, \text{ but } \mathbf{dep}(p) \vee \mathbf{dep}(p) \nvDash \mathbf{dep}(p)$$

Boolean algebras and teams

Propositional logic is the logic of Boolean algebras

$$\varphi \models \psi \text{ iff } h(\varphi) \leq h(\psi).$$

for all homomorphisms h mapping formulas into some Boolean algebra B.

- Teams are elements of $\mathcal{P}(V)$ (V is the set of valuations, $2^{\mathbb{N}}$).
- Denotations of formulas are elements of the power structure: $\mathcal{PP}(V)$, $B = \mathcal{P}(V)$.
- Logical connectives correspond to operators on $\mathcal{PP}(V)$.

Structures $\mathcal{P}(B)$ with point-wise operators have been studied under different names:

- Second-order Boolean algebras (Brink, 1984)
- Complex algebras (Goldblatt, 1989)
- Hyperboolean algebras (Goranko and Vakarelov, 1999)

Boolean algebras for teams

$\mathcal{P}(B)$ is a Boolean algebra with additional point-wise operators.

- Standard (set-theoretic) Boolean operators: $\bot$, $\neg$, $\vee$, $\wedge$
- Internal (point-wise) Boolean operators: $\bot\!\!\!\bot$, $⫬$, $\veebar$, $⩓$.

$$\begin{aligned} \bot\!\!\!\bot &= \{\, \bot \,\}\,, \\ ⫬X &= \{\, \neg a \mid a \in X \,\}\,, \\ X \mathbin{\vee\!\!\!\vee} Y &= \{\, a \vee b \mid a \in X, b \in Y \,\}\,, \text{ and} \\ X \mathbin{\wedge\!\!\!\wedge} Y &= \{\, a \wedge b \mid a \in X, b \in Y \,\}\,, \end{aligned}$$

where $X, Y \in \mathcal{P}(B)$.

Propositional Logic of Teams (LT)

Formulas of LT ($\mathtt{Fm}$):

$$\varphi ::= \bot \mid P_i \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \bot\!\!\!\bot \mid \urcorner\varphi \mid \varphi \mathbin{\vee\!\!\!\vee} \varphi \mid \varphi \mathbin{/\!\!\!\wedge} \varphi$$

Definition

$$\varphi \vDash \psi$$

iff for all Boolean algebras B and all homomorphisms $H : \mathtt{Fm} \to \mathcal{P}(B)$:

$$H(\varphi) \subseteq H(\psi).$$

$\vDash \psi$ iff $H(\psi) = B$ for all $H : \mathtt{Fm} \to \mathcal{P}(B)$.

Some remarks

- Closed under substitutions. [Composition of two homomorphisms is a homomorphism.]
- Conservative over propositional logic. [External connectives are defined over a Boolean algebra.]
- $\not\models P \veebar \neg P$ and also $\not\models P \veebar \Rsh P$ $[H(P) = \{ \bot \}]$
- $\varphi \models \psi$ not equivalent to "for all H, if $H(\varphi) = B$ then $H(\psi) = B$".
- $\perp\!\!\!\perp \vee \Rsh \perp\!\!\!\perp$ only true when $B = 2$.

Definable operators

- $\mathrm{NB} = \neg\perp\!\!\!\perp$ $\qquad B \setminus \{\, \bot \,\}$
- $\top\!\!\!\top = \urcorner\perp\!\!\!\perp$ $\qquad \{\, \top \,\}$
- $\downarrow\varphi = \varphi \mathbin{/\!\!\wedge} \top$ $\qquad \downarrow X = \{\, b \in B \mid \exists a \in X, b \leq a \,\}$
- $\uparrow\varphi = \varphi \mathbin{\vee\!\!\!\vee} \top$ $\qquad \uparrow X = \{\, b \in B \mid \exists a \in X, a \leq b \,\}$
- $\Diamond\varphi = \uparrow\downarrow\varphi$ $\qquad \Diamond X = B$ if $X \neq \emptyset$; and $\Diamond\emptyset = \emptyset$
- $\Box\varphi = \neg\Diamond\neg\varphi$ $\qquad \Box X = \emptyset$ if $X \neq B$; and $\Box B = B$

Observation

$\Box\varphi \vDash \psi$ iff for all $H : \mathbf{Fm} \to \mathcal{P}(B)$, if $H(\varphi) = B$ then $H(\psi) = B$.

Labelled formulas

Labels ($\mathtt{Lb}$) are $a ::= \bot \mid p_i \mid \neg a \mid a \vee a \mid a \wedge a$

$$a : \varphi$$

Definition

$$a : \varphi \vDash b : \psi$$

iff for all B, all $h : \mathtt{Lb} \to B$ and all $H : \mathtt{Fm} \to \mathcal{P}B$:

$$\text{if } h(a) \in H(\varphi) \text{ then } h(b) \in H(\psi).$$

Rules for external connectives

$$\frac{a : \varphi \quad a : \psi}{a : \varphi \wedge \psi}\ \wedge\mathrm{I} \qquad \frac{a : \varphi \wedge \psi}{a : \varphi}\ \wedge\mathrm{E} \qquad \frac{a : \varphi \wedge \psi}{a : \psi}\ \wedge\mathrm{E}$$

$$\frac{a : \varphi}{a : \varphi \vee \psi}\ \vee\mathrm{I} \qquad \frac{a : \psi}{a : \varphi \vee \psi}\ \vee\mathrm{I} \qquad \frac{a : \varphi \vee \psi \quad \begin{matrix}[a : \varphi] \\ \vdots \\ b : \sigma\end{matrix} \quad \begin{matrix}[a : \psi] \\ \vdots \\ b : \sigma\end{matrix}}{b : \sigma}\ \vee\mathrm{E}$$

$$\frac{\begin{matrix}[a : \varphi] \\ \vdots \\ b : \bot\end{matrix}}{a : \neg\varphi}\ \neg\mathrm{I} \qquad \frac{a : \varphi \quad a : \neg\varphi}{a : \bot}\ \neg\mathrm{E} \qquad \frac{\begin{matrix}[a : \neg\varphi] \\ \vdots \\ b : \bot\end{matrix}}{a : \varphi}\ \mathrm{RAA} \qquad \frac{a : \bot}{b : \varphi}\ \bot\mathrm{E}$$

Rules for internal connectives

$a \equiv b$ is shorthand for $a \leftrightarrow b : \mathbb{T}$.

$$\frac{a : \varphi \quad b : \psi}{a \wedge b : \varphi \mathbin{/\!\!\wedge} \psi}\ {/\!\!\wedge}\mathrm{I}$$

$$\frac{a : \varphi \quad b : \psi}{a \vee b : \varphi \mathbin{\vee\!\!\!\vee} \psi}\ {\vee\!\!\!\vee}\mathrm{I}$$

$$\dfrac{a : \varphi \mathbin{/\!\!\wedge} \psi \qquad \begin{matrix}[p : \varphi] \\ [q : \psi] \\ [a \equiv p \wedge q] \\ \vdots \\ b : \sigma\end{matrix}}{b : \sigma}\ {/\!\!\wedge}\mathrm{E}$$

$$\dfrac{a : \varphi \mathbin{\vee\!\!\!\vee} \psi \qquad \begin{matrix}[p : \varphi] \\ [q : \psi] \\ [a \equiv p \vee q] \\ \vdots \\ b : \sigma\end{matrix}}{b : \sigma}\ {\vee\!\!\!\vee}\mathrm{E}$$

$$\frac{a : \varphi}{\neg a : \urcorner\varphi}\ \urcorner\mathrm{I} \qquad \frac{a : \urcorner\varphi}{\neg a : \varphi}\ \urcorner\mathrm{E}$$

Rules for labels

$$\frac{a_1 : \mathbb{T} \quad \cdots \quad a_k : \mathbb{T}}{b : \mathbb{T}} \; \texttt{taut}$$

The rule `taut` is only applicable if $a_1, \ldots, a_k \vdash b$ in classical propositional logic, i.e., if $a_1 \wedge \ldots \wedge a_k \to b$ is a tautology.

$$\frac{a \equiv b \quad b : \varphi}{a : \varphi} \; \texttt{sub}$$

Results

Observation

$$\varphi \vDash \psi \quad \text{iff} \quad p : \varphi \vDash p : \psi$$

Soundness and completeness

$$a : \varphi \vDash b : \psi \quad \text{iff}$$

$$a : \varphi \vdash b : \psi$$

Theorem

- The set of countable (incl finite) Boolean algebras is adequate for LT.
- The set of finite Boolean algebras is not adequate.

Strict negation

- $\sim\varphi = \neg\uparrow(\downarrow\varphi \wedge \neg\perp\!\!\!\perp)$
- $\sim X = \{\, b \in B \mid \forall a \in X, b \wedge a = \bot \,\}$
- $\sim\top\!\!\!\top = \perp\!\!\!\perp$
- $\not\models P \vee\!\!\!\vee \sim P$

Strong propositional team logic PT^+

Yang and Väänänen (2017)

$$\varphi ::= P_i \mid {\sim}P_i \mid \perp\!\!\!\perp \mid \mathrm{NB} \mid \varphi \mathbin{\vee\!\!\!\vee} \varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi$$

$H_V : \mathbf{Fm}_{\mathrm{PT}^+} \to \mathcal{P}\mathcal{P}(2^{\mathbb{N}})$

$$H_V(P_i) = \{\, a \in \mathcal{P}(2^{\mathbb{N}}) \mid \forall v \in a, v(i) = 1 \,\}$$
$$H_V({\sim}P_i) = \{\, a \in \mathcal{P}(2^{\mathbb{N}}) \mid \forall v \in a, v(i) = 0 \,\}$$

[Note that $H_V(P_i)$ is a principal ideal.]

$$\mathrm{PT}^+ : \varphi \vDash \psi \quad \text{iff} \quad H_V(\varphi) \subseteq H_V(\psi)$$

Axiomatizing PT^+ in LT

Theorem

$X \Vvee {\sim}X = B$ iff X is a principal ideal.

$\mathsf{PVA} = \{\, \Box(P_i \Vvee {\sim}P_i) \mid i \in \mathbb{N} \,\}$.

Theorem

$$\mathsf{PT}^+ : \varphi \vDash \psi \quad \text{iff}$$

$$\mathsf{LT} : \mathsf{PVA}, \varphi \vDash \psi.$$

Further work

- What are the algebras of LT? Is there a representation theorem? More specifically are there algebras that satisfy all LT-equations but is not isomorphic to a substructure of a $\mathcal{P}(B)$?
- The semantical construction of LT can be seen as “adding” a Boolean algebra on a another Boolean algebra, and the proof system is clearly propositional logic “on top” of propositional logic. Are similar constructions applicable to other logics?
- Specifically, what is propositional logic “on top” of first-order logic?

THANKS!

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