Team semantics and substitutionality

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joint work with Orvar Lorimer Olsson

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Nordic Online Logic seminar

The plan

- Background and motivation
  - Standard team semantics and flatness
  - Non-standard team semantics: Generalized quantifiers
  - Substitutionality
- A substitutional propositional logic of teams
  - Semantics and adequacy
  - Deductions and completeness
  - Axiomatizing standard team semantics
- Further work

Background and motivation

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- Dependence logic = First-order logic + dependence atom.

Dependence atom

	x	y	z	w
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Definition

$M, X \models \mathsf{dep}(\bar{x}, y)$ iff for all $s, s' \in X$ if $s(\bar{x}) = s'(\bar{x})$ then $s(y) = s'(y)$.

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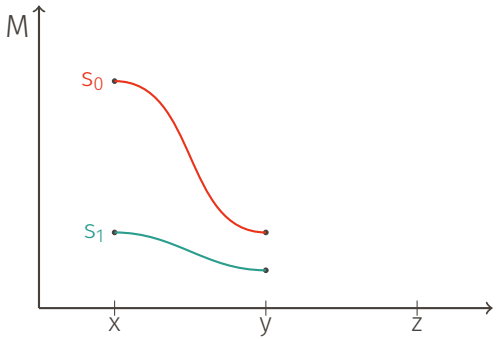

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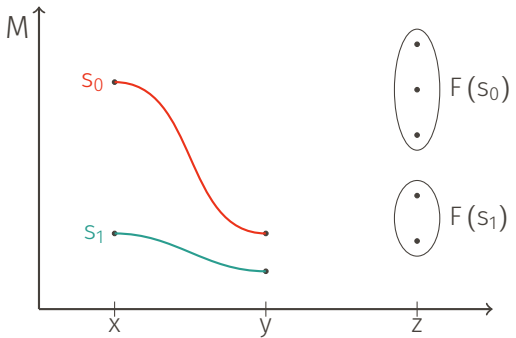

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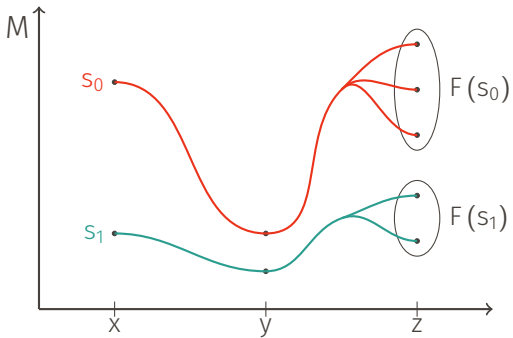

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- **Flatness:** For FO-formulas: $M, X \models \varphi$ iff $M, s \models \varphi$ for all $s \in X$.
- Dependence logic and Existential second-order logic (ESO/Σ^1_1) are equivalent.

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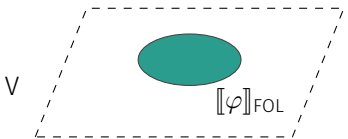

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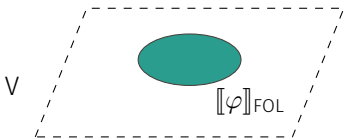

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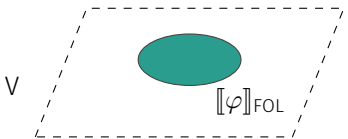

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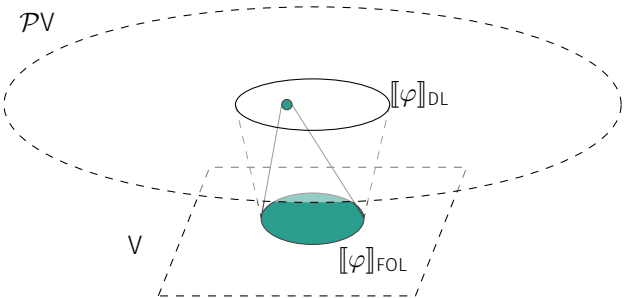

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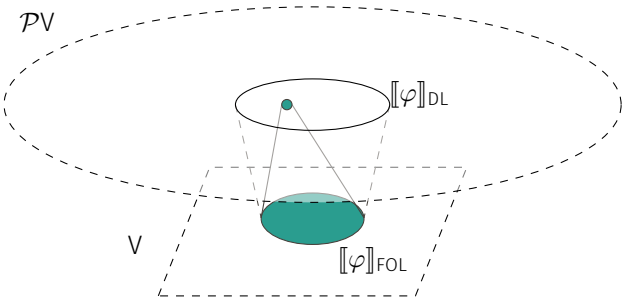

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$$[\![\varphi]\!]_{DL} = \mathcal{P}([\![\varphi]\!]_{FOL})$$

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$$M, X \models Qx\varphi \text{ iff there is } F : X \to Q_M \text{ such that } M, X[F/x] \models \varphi.$$

- $Q_M = \{ P \mid (M, P) \in Q \}$

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Theorem (Engström (2024))

Using this alternative principle we can define a non-standard team semantics that can handle any generalized quantifiers.

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Tarski (1936), property (F)

If, in the sentences of the class K and in the sentence X, the constants— apart from purely logical constants—are replaced by any other constants (like signs being everywhere replaced by like signs), and if we denote the class of sentences thus obtained from K by 'K'', and the sentence obtained from X by 'X'', then the sentence X′ must be true provided only that all sentences of the class K′ are true.

Dependence logic and substitutionality

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Due to that the possible denotations of atoms is a proper subset of the possible denotations of formulas.

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Basic question

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- Start with propositional logic, for simplicty and its strong connection to Boolean algebras.
- Make sure that the denotations of propositional variables are as general as possible.
- Include “enough” connectives to be able to express standard team semantics.

Propositional logic of teams

Boolean algebras and teams

Propositional logic is the logic of Boolean algebras

$$\varphi \models \psi \text{ iff } \mathsf{h}(\varphi) \leq \mathsf{h}(\psi).$$

for all Boolean algebras B and homomorphisms $\mathsf{h} : \mathtt{Fm} \to \mathsf{B}$.

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Structures $\mathcal{P}(\mathsf{B})$ with point-wise operators have been studied under different names:

- Second-order Boolean algebras (Brink, 1984)
- Complex algebras (Goldblatt, 1989)
- Hyperboolean algebras (Goranko and Vakarelov, 1999)

Algebras for teams

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$$
\begin{aligned}
⫫ &= \{ \bot \}, \\
\dashv X &= \{ \neg a \mid a \in X \}, \\
X \mathbin{\mathbb{V}} Y &= \{ a \vee b \mid a \in X, b \in Y \}, \text{ and} \\
X \mathbin{⩚} Y &= \{ a \wedge b \mid a \in X, b \in Y \},
\end{aligned}
$$

where $X, Y \in \mathcal{P}(B)$.

Propositional Logic of Teams (LT)

Formulas of LT (**Fm**):

$$\varphi ::= \bot \mid P_i \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \perp\!\!\!\perp \mid \Box\varphi \mid \varphi \mathbin{\mathbb{V}} \varphi \mid \varphi \mathbin{/\!\!/\!\wedge} \varphi$$

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$\vDash \psi$ iff $\mathrm{H}(\psi) = \mathrm{B}$ for all $\mathrm{H} : \mathtt{Fm} \to \mathcal{P}(\mathrm{B})$.

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LT is closely related to "Hyperboolean modal logic" (HBML) of Goranko and Vakarelov (1999).

Labelled formulas

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Rules for external connectives

$$\frac{\mathsf{a} : \varphi \quad \mathsf{a} : \psi}{\mathsf{a} : \varphi \wedge \psi}\ \wedge\mathsf{I} \qquad \frac{\mathsf{a} : \varphi \wedge \psi}{\mathsf{a} : \varphi}\ \wedge\mathsf{E} \qquad \frac{\mathsf{a} : \varphi \wedge \psi}{\mathsf{a} : \psi}\ \wedge\mathsf{E}$$

$$\frac{\mathsf{a} : \varphi}{\mathsf{a} : \varphi \vee \psi}\ \vee\mathsf{I} \qquad \frac{\mathsf{a} : \psi}{\mathsf{a} : \varphi \vee \psi}\ \vee\mathsf{I} \qquad \frac{\mathsf{a} : \varphi \vee \psi \qquad \begin{array}{c}[\mathsf{a} : \varphi] \\ \vdots \\ \mathsf{b} : \sigma\end{array} \qquad \begin{array}{c}[\mathsf{a} : \psi] \\ \vdots \\ \mathsf{b} : \sigma\end{array}}{\mathsf{b} : \sigma}\ \vee\mathsf{E}$$

$$\frac{\begin{array}{c}[\mathsf{a} : \varphi] \\ \vdots \\ \mathsf{b} : \bot\end{array}}{\mathsf{a} : \neg\varphi}\ \neg\mathsf{I} \qquad \frac{\mathsf{a} : \varphi \quad \mathsf{a} : \neg\varphi}{\mathsf{a} : \bot}\ \neg\mathsf{E} \qquad \frac{\begin{array}{c}[\mathsf{a} : \neg\varphi] \\ \vdots \\ \mathsf{b} : \bot\end{array}}{\mathsf{a} : \varphi}\ \mathsf{RAA} \qquad \frac{\mathsf{a} : \bot}{\mathsf{b} : \varphi}\ \bot\mathsf{E}$$

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$$\frac{a_1 : \mathbb{T} \quad \cdots \quad a_k : \mathbb{T}}{b : \mathbb{T}}\ \texttt{taut}$$

Assuming $a_1, \ldots, a_k \vdash b$ in propositional logic, i.e., that $a_1 \wedge \ldots \wedge a_k \to b$ is a tautology.

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Soundness and completeness

$$a : \varphi \vDash b : \psi \quad \text{iff} \quad a : \varphi \vdash b : \psi$$

Strong propositional team logic: PT^+

Yang and Väänänen (2017)

$$\varphi ::= \mathrm{P}_i \mid {\sim}\mathrm{P}_i \mid \perp\!\!\!\perp \mid \mathrm{NB} \mid \varphi \mathbin{\mathbb{V}} \varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi$$

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Note that $\mathrm{H}_V(\mathrm{P}_i)$ are principal ideals.

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- $\Box\varphi = \neg\Diamond\neg\varphi$ — $\Box X = \emptyset$ if $X \neq B$; and $\Box B = B$

Observation

$\Box\varphi \vDash \psi$ iff for all $H : \mathsf{Fm} \to \mathcal{P}(B)$, if $H(\varphi) = B$ then $H(\psi) = B$.

- $\sim\varphi = \neg\uparrow(\downarrow\varphi \wedge \neg\!\perp\!\!\!\perp)$ — $\sim X = \{\, b \in B \mid \forall a \in X, b \wedge a = \bot \,\}$

Note that $\not\vDash P \mathbin{\backslash\!\!\backslash} \sim P$.

Axiomatizing PT^+ in LT

Theorem

$X \mathbin{\mathbb{V}} {\sim}X = B$ iff X is a principal ideal.

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Axiomatizing PT^+ in LT

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Theorem

$\mathrm{PT}^+ : \varphi \vDash \psi \quad$ iff $\quad \mathrm{LT} : \mathrm{PVA}, \varphi \vDash \psi.$

Conclusion

There is a substitutional propositional logic with a complete labelled natural deduction system that can express everything that standard propositional dependence logics can.

Further work

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- The proof system can be seen as propositional logic “on top” of propositional logic. Are similar constructions applicable to other logics?
- Specifically, what is propositional logic “on top” of first-order logic?

Further work: First-order logic

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It seems that $\mathrm{PVA} = \{\, \Box(P_i \mathbin{\backslash\!\!\backslash} {\sim}P_i) \mid i \in \mathbb{N} \,\}$ axiomatizes standard team semantics in this setting as well.

THANKS!

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