Generalized quantifiers and team semantics

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$$\forall x \exists y \forall z \exists w\, Rxyzw$$

$$\downarrow$$

$$\forall z$$

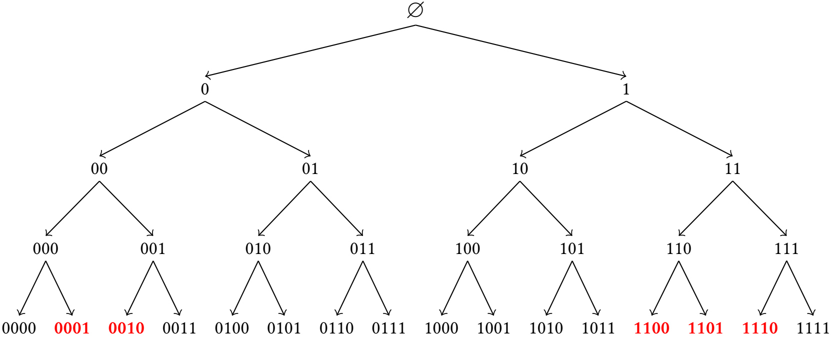
$$\downarrow$$

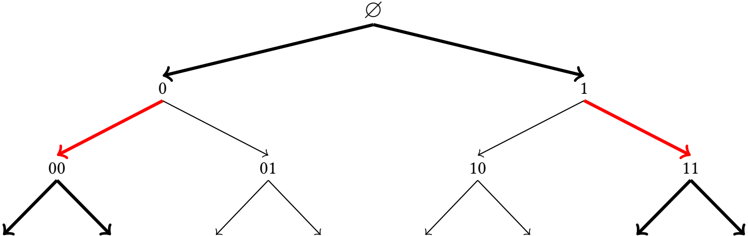
$$\exists w$$

$$\begin{pmatrix} \forall x \exists y \\ \forall z \exists w \end{pmatrix} Rxyzw$$

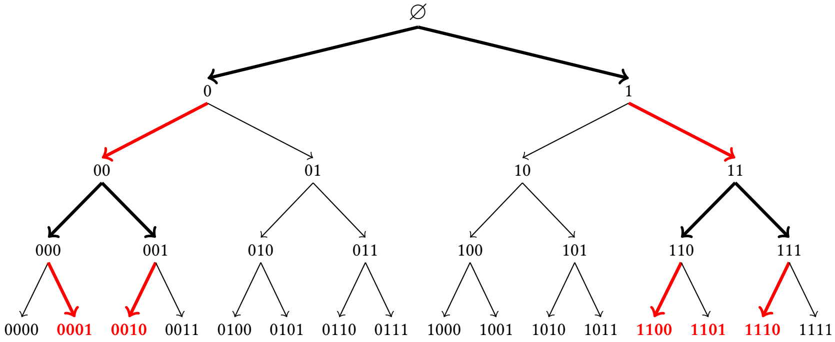
$$\begin{array}{cc} \forall z & \forall x \\ \downarrow & \downarrow \\ \exists w & \exists y \end{array}$$

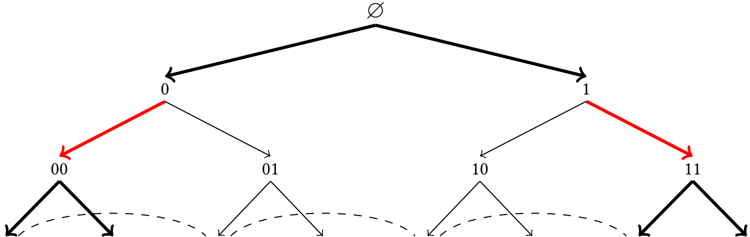
Domain $\{0, 1\}$. $\forall x \exists y \forall z \exists w Rxyzw$





Domain { 0, 1 }. $\begin{pmatrix} \forall x \exists y \\ \forall z \exists w \end{pmatrix}$ *Rxyzw*





x	y	z	w
0	0	0	1
0	0	1	0
1	1	0	0
1	1	1	0

$$\not\models D(z; w)$$

x	y	z	w
0	0	0	1
0	0	1	0
1	1	0	1
1	1	1	0

$$\models D(z; w)$$

$$\begin{pmatrix} \forall x \exists y \\ \forall z \exists w \end{pmatrix} Rxyzw \equiv \forall x \exists y \forall z \exists w \left(D(z, w) \wedge Rxyzw \right)$$

Definition

A **team** X is a set of variable assignments (with the

Team semantics

- **Team semantics:** Lifts semantic values (of formulas) from sets of assignment to sets of sets of assignments (or sets of **teams**).
- **Flatness for FO:** A first-order formula is satisfied by a team iff all assignments in the team satisfy the formula.

Dependence Logic

$$\phi ::= \mathrm{At} \mid \neg\mathrm{At} \mid D(\bar{x}; y) \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x \phi \mid \forall x \phi$$

- $M, X \models \gamma$ if for all $s \in X$: $M, s \models \gamma$, where γ is a literal.
- $M, X \models D(\bar{x}; y)$ if for all $s, s' \in X$ if $s(\bar{x}) = s'(\bar{x})$ then $s(y) = s'(y)$.
- $M, X \models \phi \wedge \psi$ if $M, X \models \phi$ and $M, X \models \psi$.
- $M, X \models \phi \vee \psi$ if there are $Y \cup Z = X$ s.t. $M, Y \models \phi$ and $M, Z \models \psi$.
- $M, X \models \exists x \phi$ if there is $f : X \to M$ s.t. $M, X[f/x] \models \phi$.
- $M, X \models \forall x \phi$ if $M, X[M/x] \models \phi$.

- $X[f/x] = \{ s[f(s)/x] \mid s \in X \}$ ($s[a/x] = s \cup \langle x, a \rangle$)
- $X[M/x] = \{ s[a/x] \mid s \in X \text{ and } a \in M \}$

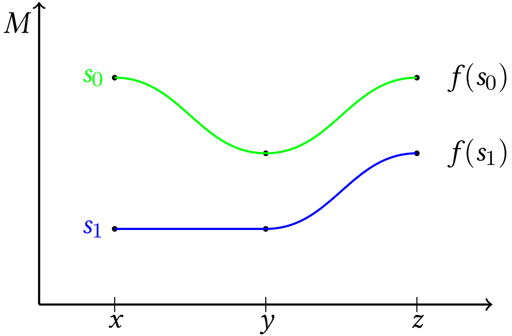
$$M \models \sigma \text{ iff } M, \{\emptyset\} \models \sigma.$$

Dependence Logic, quantifiers

- $M, X \models \exists x\, \phi$ if there is $f : X \to M$ s.t. $M, X[f/x] \models \phi$.

- $X[f/x] = \{\, s[f(s)/x] \mid s \in X \,\}$

Example: $M, \{s_0, s_1\} \models \exists z\, Rxyz$

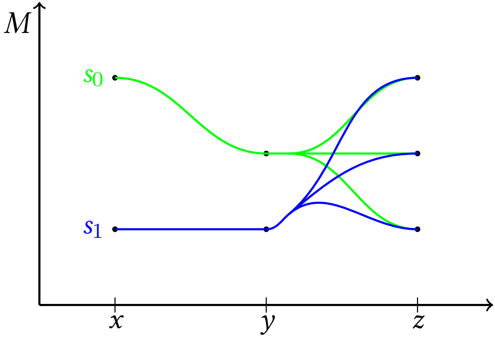


Dependence Logic, quantifiers

- $M, X \models \forall x\, \phi$ if $M, X[M/x] \models \phi$.

- $X[M/x] = \{\, s[a/x] \mid s \in X \text{ and } a \in M \,\}$

Example: $M, \{s_0, s_1\} \models \forall z\, Rxyz$



Examples

$$\forall x \exists y \forall z \exists w \big(D(z; w) \wedge \phi \big)$$

$$\exists u \forall x \exists y \forall z \exists w \big(D(z; w) \wedge \neg u = y \wedge (x = z \leftrightarrow y = w) \big)$$

Properties

- **Empty team property:** $M, \emptyset \models \phi$
- **Downwards closure:** If $Y \subseteq X$ and $M, X \models \phi$ then $M, Y \models \phi$.
- Dependence logic (DL) $\equiv$ Existential Second Order logic (ESO)
- For formulas the situation is slightly different: Only a fragment of ESO is expressible in DL.
- **Extra special feature of DL:** Truth is definable.

Branching in natural languages

Some relative of each villager and some relative of each townsmen hate each other. (Hintikka, 1974)

$$\begin{pmatrix} \forall x \exists y \\ \forall z \exists w \end{pmatrix} (V(x) \wedge T(z) \rightarrow (R(x,y) \wedge R(z,w) \wedge H(y,w)))$$

Most of the dots and most of the stars are all connected by lines. (Barwise, 1979)

$$\begin{pmatrix} Q_1 x \\ Q_2 y \end{pmatrix} C(x,y)$$

Two examiners marked six scripts. (Davies, 1989)

$$\begin{pmatrix} \exists^{\geqslant 2} x \\ \exists^{\geqslant 6} y \end{pmatrix} E(x) \wedge S(y) \wedge M(x,y)$$

Generalized quantifiers

A generalized quantifier Q is a class of structures closed under isomorphisms.

- $Q_M := \{ R \mid (M, R) \in Q \}$.

$$Q_M \subseteq \mathcal{P}(M).$$

$$M, s \models Qx\,\phi \text{ iff } [\![\phi]\!]^{M,s} \in Q_M$$

- $\forall_M = \{ M \}$
- $\exists_M = \{ A \subseteq M \mid A \neq \emptyset \}$
- $(Q_1)_M = \{ A \subseteq M \mid |A| \geqslant \aleph_1 \}$

Q is **monotone increasing** if $A \subseteq B$ and $A \in Q_M$ implies $B \in Q_M$.

Generalized quantifiers in dependence logic

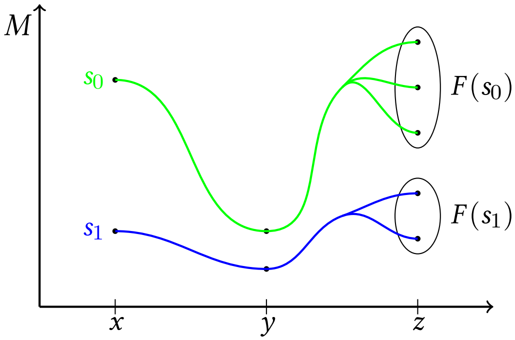
(Engström, 2012)

Works well only for **monotone increasing generalized quantifiers**.

- $M, X \models Qx\,\phi$ iff there is $F : X \to Q_M$ such that $M, X[F/x] \models \phi$.

$$X[F/x] = \{\, s[a/x] \mid s \in X, a \in F(s) \,\}$$

Example: $M, \{s_0, s_1\} \models \exists^{\geqslant 2} z\, Rxyz$



Iteration and branching

Iteration

$(Q_1 \cdot Q_2)_M = \{ R \subseteq M^2 \mid \{ a \mid {}_aR \in (Q_2)_M \} \in (Q_1)_M \}$

$$(Q_1 \cdot Q_2)xy\phi \equiv Q_1 x\, Q_2 y\, \phi$$

For monotone increasing quantifiers:

$\mathrm{Br}(Q_1, Q_2)_M = \{ R \subseteq M^2 \mid A \times B \subseteq R, A \in (Q_1)_M, B \in (Q_2)_M \}$

$$\mathrm{Br}(Q_1, Q_2)xy\phi \equiv \begin{pmatrix} Q_1 x \\ Q_2 y \end{pmatrix} \phi$$

Properties of DL(Q)

- **Empty team property:** $M, \emptyset \models \phi$
- **Downwards closure:** If $Y \subseteq X$ and $M, X \models \phi$ then $M, Y \models \phi$.

Flatness

$$M, X \models \phi \text{ iff for all } s \in X : M, s \models \phi$$

for all FO(Q)-formulas ϕ.

Iteration

$$M, X \models (Q_1 \cdot Q_2)xy\phi \text{ iff } M, X \models Q_1 x Q_2 x \phi$$

Branching

$$\mathrm{DL}(Q) \equiv \mathrm{DL}(Q, \mathrm{Br}(Q, Q))$$

Strength and axiomatizability

Theorem (Engström and Kontinen, 2013)

$$\mathrm{DL}(Q) \equiv \mathrm{ESO}(Q)$$

Let $\Gamma \models_w \phi$ mean that $\Gamma \models \phi$ for any monotone increasing interpretation of Q.

Theorem (Engström et al., 2017)
There are sound and complete inference systems wrt the following consequence relations:

- $\Gamma \models_w \phi$ where ϕ is $\mathrm{FO}(Q)$.
- $\Gamma \models \phi$ where ϕ is $\mathrm{FO}(Q_1)$.

Digression: Questions

"Who won Tour de France this year?" $\models$ "This year's Tour de France has finished."

Inquisitive semantics (Ciardelli, 2016; Yang and Väänänen, 2016)

- Information states X are sets of models (together with variable assignments).
- $X \models \phi$ if for all $M \in X$, $M \models \phi$ (when ϕ first-order).
- $X \models \phi \vee\!\!\!\vee \psi$ if $X \models \phi$ or $X \models \psi$
- $?\phi := \phi \vee\!\!\!\vee \neg\phi$

The inquisitive semantics is very close to the semantics of dependence logic, in fact:

$$D(x; y) \text{ is equivalent to } \lambda_x \to \lambda_y.$$

λ_x is the **identity question** about x, "What is the value of x?".

Non-monotone quantifiers

$$M \models \exists^{=5} x\, Px$$

$$\exists F : \{\varnothing\} \to \exists^{=5}_{M},\ \text{s.t. } M, \{\varnothing\}[F/x] \models Px$$

$$\exists A \subseteq M,\ \text{s.t. } |A| = 5 \text{ and } A \subseteq P^{M}$$

$$M \models \exists^{\geqslant 5} x\, Px$$

ϕ is satisfied by X if

- every assignment $s \in X$ satisfies ϕ.
- ~~every assignment $s \in X$ satisfies ϕ.~~
- for every assignment $s : \mathrm{dom}(X) \to M^k$, $s \in X$ **iff** s satisfies ϕ.

Team logic

$$\phi ::= \mathrm{At} \mid \neg\mathrm{At} \mid \top(\bar{x}) \mid \phi \otimes \phi \mid \phi \oplus \phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x \phi \mid \forall x \phi$$

- $M, X \models \psi$ iff $\forall s : \mathrm{dom}(X) \to M(s \in X$ iff $M, s \models \psi)$, for first-order atomic or negated atomic formulas ψ.
- $M, X \models \top(\bar{x})$ iff $\exists \bar{x} X = \{ \epsilon \} [M^k / \mathrm{dom}(X) \setminus \{ \bar{x} \}]$
- $M, X \models \phi \otimes \psi$ iff $\exists Y, Z$ s.t. $X = Y \cap Z$; $M, Y \models \phi$ and $M, Z \models \psi$
- $M, X \models \phi \oplus \psi$ iff $\exists Y, Z$ s.t. $X = Y \cup Z$; $M, Y \models \phi$ and $M, Z \models \psi$
- $M, X \models \phi \wedge \psi$ iff $M, X \models \phi$ and $M, X \models \psi$
- $M, X \models \phi \vee \psi$ iff $M, X \models \phi$ or $M, X \models \psi$
- $M, X \models \exists x \phi$ iff $\exists Y$ s.t. $x \in \mathrm{dom}(Y), \exists x Y = \exists x X$ and $M, Y \models \phi$
- $M, X \models \forall x \phi$ iff $\exists Y$ s.t. $x \in \mathrm{dom}(Y), \forall x Y = \exists x X$ and $M, Y \models \phi$

$$Qx\, X = \{ s : \mathrm{dom}(X) \setminus \{ x \} \to M \mid \{ a \in M \mid s[a/x] \in X \} \in Q_M \}$$

Properties

Basic principle

A formula ϕ is satisfied by a team X if for every assignment $s : \mathrm{dom}(X) \to M^k$, $s \in X$ iff s satisfies ϕ, i.e.,

$$M, X \models \phi \text{ iff } X = [\![\phi]\!]^M_{\mathrm{dom}(X)} .$$

A formula is **untangled** if no quantifier Qx appears in the scope of another quantifier $Q'x$ and no variable is both free and bound.

Theorem

For first-order untangled ϕ and teams X s.t. $\mathrm{dom}(X) \cap \mathrm{bv}(\phi) = \varnothing$:

$$M, X \models \phi \text{ iff } X = [\![\phi]\!]^M_{\mathrm{dom}(X)}$$

Relationship with Dependence Logic

$X \models D(\bar{x}; y)$ iff

$$X \models \exists z \big(\forall \bar{w} (\top(\bar{x}, y) \otimes \top(\bar{x}, z)) \wedge (y = z \otimes \top(\bar{x}, \bar{w}))\big),$$

where z is not in $\bar{x}, y$ and $\bar{w}$ is $\mathrm{dom}(X) \setminus \{ \bar{x}, y, z \}$.

Theorem

For every team X and formula ϕ of Dependence logic such that $\mathrm{dom}(X) = \mathrm{fv}(\phi)$ there is ϕ^+ of team logic:

$$M, X \models_{\mathrm{DL}} \phi \text{ iff } M, X \models_{\mathrm{TL}} \phi^+.$$

Team logic $\equiv$ ESO

Generalized quantifiers revisited

Definition

$M, X \models Q\bar{x}\phi$ if there is Y such that $\bar{x} \in \text{dom}(Y)$, $M, Y \models \phi$ and $\exists\bar{x}X = Q\bar{x}Y$, where

$$Q\bar{x}Y = \{\, s : \text{dom}(Y) \setminus \{\, \bar{x} \,\} \to M \mid Y_s(\bar{x}) \in Q_M \,\}\,.$$

$Y_s = \{\, s' : \text{dom}(Y) \setminus \text{dom}(s) \to M \mid s \cup s' \in Y \,\}$.

Flatness

For every untangled ϕ formula of $\text{FO}(Q)$ and every team X such that $\text{dom}(X) \cap \text{bv}(\phi) = \varnothing$:

$$M, X \models \phi \text{ iff } X = [\![\phi]\!]^M_{\text{dom}(X)}\,.$$

Iteration

$$M, X \models (Q_1 \cdot Q_2)xy\phi \;\text{ iff }\; M, X \models Q_1 x\, Q_2 x\, \phi$$

Take home message

Lifting from Tarskian semantics to team semantics makes it possible to logically analyse phenomena in natural languages such as branching, questions and dependence.

That's all folks!

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