Resplendency and Omitting Types

Work in progress

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Question

Setup:

- $\mathcal{L}$ — A recursive language.
- M — An $\mathcal{L}$-structure.
- $\mathcal{L}^+$ — A recursive extension of $\mathcal{L}$.
- T — A theory in $\mathcal{L}^+$.
- $p(x)$ a (semi-)type in $T + \mathrm{Th}(M)$.

Question: When does there exist an expansion of M to $\mathcal{L}^+$ satisfying T and omitting $p(x)$.

Notation

- **Def:** A *(semi-)type* $p(x)$ in a model M is a set of formulas with a finite number of parameters from M s.t. $\mathrm{Th}(M, a)_{a \in M} + p(c)$ is consistent.
- **Def:** A type $p(x)$ is *complete* if $p(c)$ is complete.
- **Def:** We will write $p\!\uparrow$ for the $\mathcal{L}^{+}_{\omega_1\omega}$-sentence

$$\forall x \bigvee_{\psi(x) \in p(x)} \neg\psi(x)$$

saying that $p(x)$ is omitted, and $p\!\downarrow$ for

$$\exists x \bigwedge p(x)$$

saying that $p(x)$ is realized.

Recursive saturation

- **Def:** M is *recursively saturated* if every recursive type in M is realized.
- **Def:** A *Scott set* is a subset of $\mathcal{P}(\mathbb{N})$ closed under union, complement, relative recursiveness and König's lemma (every infinte binary tree has an infinite path).
- **Def:** Given a Scott set $\mathfrak{X}$, M is $\mathfrak{X}$-saturated if

$$\mathfrak{M} \models p(\bar{x}, \bar{a})\downarrow \quad \text{iff} \quad p(\bar{x}, \bar{y}) \in \mathfrak{X}$$

  for every complete type $p(\bar{x}, \bar{a})$.
- Every countable recursively saturated model M is $\mathfrak{X}$-saturated for some countable Scott set $\mathfrak{X}$.
- Every $\mathfrak{X}$-saturated model is recursively saturated.

Resplendency

- **Def:** A model M is *resplendent* if for every recursive extension $\mathcal{L}^+$ of $\mathcal{L}$ and every recursive T in $\mathcal{L}^+$ such that $\mathrm{Th}(M) + T$ is consistent there is an expansion M^+ of M satisfying T.
- Every countable recursively saturated model is resplendent, and every resplendent model is recursively saturated.

Thus, for countable M, resplendency is in fact a saturation property.

Arithmetical saturation

- **Def:** A model is *arithmetically saturated* if it is $\mathfrak{X}$-saturated for a Scott set $\mathfrak{X}$ closed under the jump operation (or equivalenty closed under Σ_1 comprehension).
- There is a recursive theory T and a recursive type $p(x)$ (in T) such that a countable recursively saturated model of PA is arithmetically saturated iff there is an expansion M^+ of M such that $M^+ \models T + p\uparrow$ (see [Kaye et al.(1991)Kaye, Kossak, and Kotlarski]).
- T expresses that g is an automorphism, and $p\uparrow$ expresses that g moves all non-definable points.

ω-saturation

Theorem 1. *If M is a countable (ω-)saturated model then $X(T,p)$ holds for every T and p such that $C_1(T,p)$ holds.*

- **Def:** $X(T,p)$ is 'there exists an expansion of M satisfying T and omitting p.'
- **Def:** $C_1(T,p)$ is 'there exists an ω-homogeneous model $N \succ M$ and an expansion N^+ of N such that $N^+ \models T + p\uparrow$.'

Analytic saturation

Theorem 2. *If $M \models$ PA is such that $X(T,p)$ holds for every recursive T and p such that $C_2(T,p)$ then M is analytical saturated.*

- **Def:** $C_2(T,p)$ is ‘there exists an ω-saturated model $N \models \mathrm{Th}(M)$ and an expansion N^+ of N such that $N^+ \models T + p\!\uparrow$.’
- **Def:** A model is *analytical saturated* if it is $\mathfrak{X}$-saturated for a Scott set $\mathfrak{X}$ closed under the hyperjump operation (or equivalenty a β-model closed under Σ^1_1 comprehension).

More saturation

Theorem 3. *Let M be countable and $\mathfrak{X}$-saturated. If $\mathfrak{X}$ is closed under completion in C_2 then for every $T, p \in \mathfrak{X}$*

$$C_2(T,p) \Rightarrow X(T,p).$$

- **Def:** A scott set $\mathfrak{X}$ is closed under completion in C_2 if for every $T, p \in \mathfrak{X}$ such that $C_2(T,p)$ holds there exists a completion $T^C \in \mathfrak{X}$ of T such that $C_2(T^C, p)$ holds.
- It is easy to see that for any countable Scott set $\mathfrak{X}$ there is another countable Scott set $\mathfrak{Y}$ such that $\mathfrak{Y}$ is closed under completion in C_2 and $\mathfrak{X} \subseteq \mathfrak{Y}$.

Extra

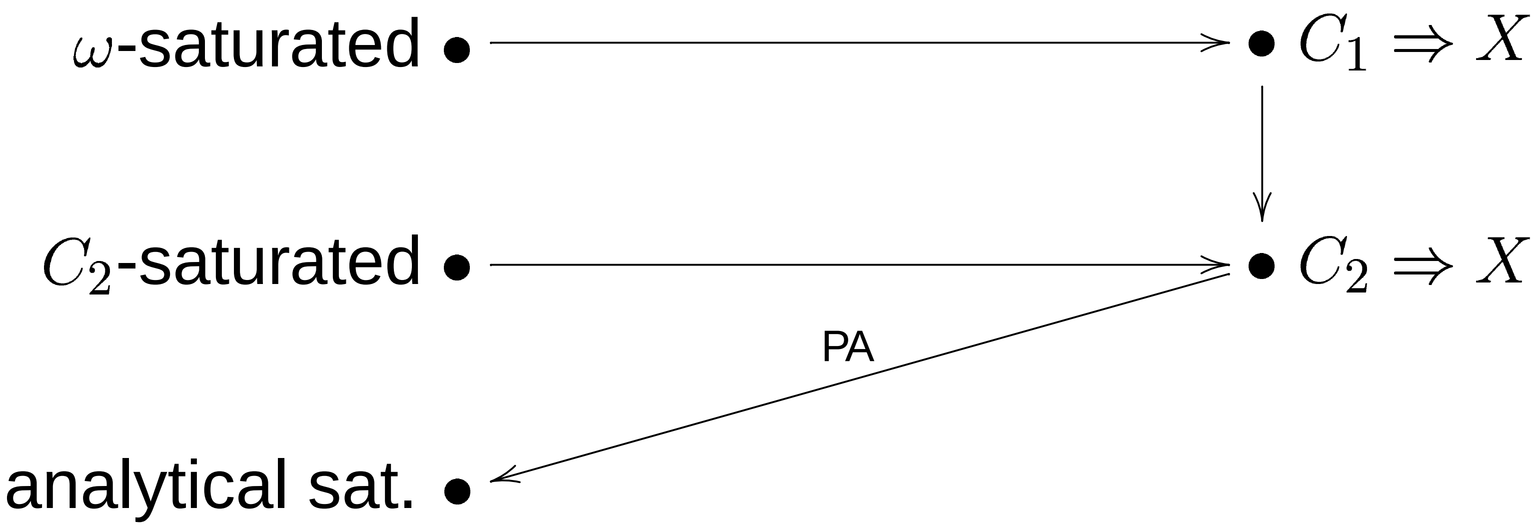
Theorem 4. *If a countable model M is analytical saturated then for every $T, p \in \mathfrak{X}$ (M is $\mathfrak{X}$-saturated) such that $C_3(T, p)$ holds there exists $N \prec M$ and an expansion N^+ of N such that $N^+ \models T + p\!\uparrow$.*

- **Def:** $C_3(T, p)$ is 'there is model of $\mathrm{Th}(M) + T + p\!\uparrow$.'

Theorem 5. *If M is a model of PA such that for every recursive T, p satisfying C_3 there exists $N \prec M$ and an expansion N^+ of N such that $N^+ \models T, p$ then $\mathrm{SSy}(M)$ is a β-model.*

- **Def:** A set $\mathfrak{X} \subseteq \mathcal{P}(\mathbb{N})$ is a β-model if it models all Σ^1_1-sentences (in the language of arithmetic) true in $\mathbb{N}$.

Summary

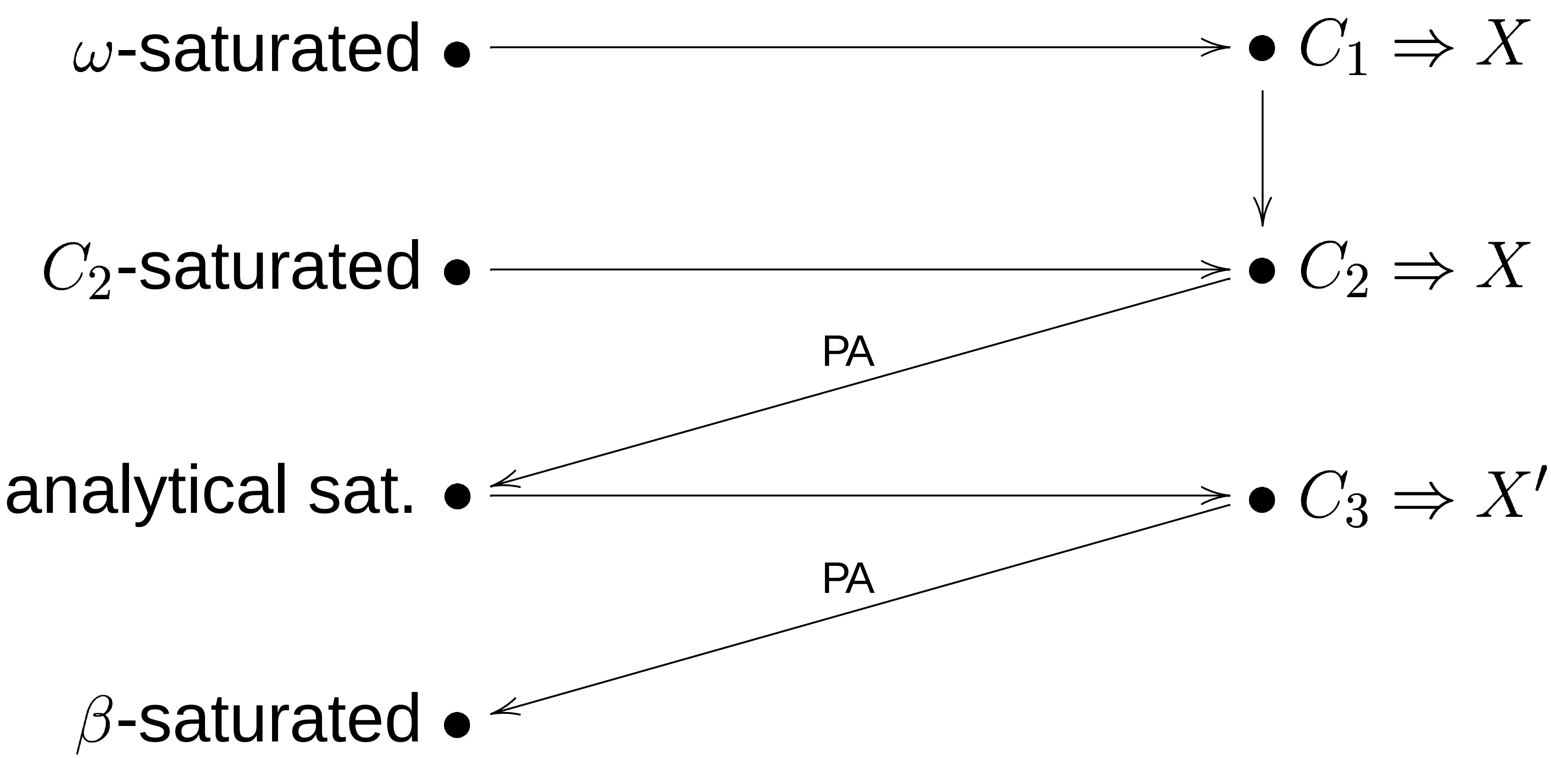


- $C_i \Rightarrow X$ means 'for all recursive extensions $\mathcal{L}^+$ and all recursive T and $p(x)$ if $C_i(T,p)$ holds then $X(T,p)$ holds.

Saturation and consistency notions

- ω**-sat:** all types are realized.
- C_2**-sat:** $\mathfrak{X}$-saturated for $\mathfrak{X}$ closed under completion in C_2.
- **ana.sat.:** $\mathfrak{X}$-saturated for $\mathfrak{X}$ closed under analytical comprehension.
- C_1**:** exists ω-homomgeneous $N \succ M$ and expansion N^+ of N s.t. $N \models T + p\uparrow$.
- C_2**:** exists ω-saturated $N \succ M$ and expansion N^+ of N s.t. $N \models T + p\uparrow$

Summary (alt)



- $C_i \Rightarrow X$ is 'for all recursive extensions $\mathcal{L}^+$, all recursive T and $p(x)$ if $C_i(T,p)$ then $X(T,p)$ holds.
- $X'(T,p)$ is 'there exists $N^+ \models T + p{\uparrow}$ such that $N^+ \prec_{\mathcal{L}} M$.

Saturation and consistency notions (alt)

- ω**-sat:** all types are realized.
- C_2**-sat:** $\mathfrak{X}$-saturated for $\mathfrak{X}$ closed under completion in C_2.
- **ana.sat.:** $\mathfrak{X}$-saturated for $\mathfrak{X}$ closed under analytical comprehension.
- β**-sat:** $\mathfrak{X}$-saturated for $\mathfrak{X}$ s.t. all true Σ^1_1 sentences are true in $\mathfrak{X}$.
- C_1: exists ω-homomgeneous $N \succ M$ and expansion N^+ of N s.t. $N \models T + p\uparrow$.
- C_2: exists ω-saturated $N \succ M$ and expansion N^+ of N s.t. $N \models T + p\uparrow$
- C_3: exists N s.t. $N \models \mathrm{Th}(M) + T + p\uparrow$.

References

References

[Kaye et al.(1991)Kaye, Kossak, and Kotlarski] Richard Kaye, Roman Kossak, and Henryk Kotlarski. Automorphisms of recursively saturated models of arithmetic. *Ann. Pure Appl. Logic*, 55(1):67–99, 1991. ISSN 0168-0072.