Is dependence logical?

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My background

Dependence logic

Logical constants

My background

- Non-standard models of Peano Arithmetic, PA.
- Extending the notion of resplendent models to non-first-order languages (transplendent models).
- Scott set: A boolean algebra of sets of natural numbers closed under computability and weak Königs lemma.
- Standard system: $\mathrm{SSy}(M) = \{ A \cap \mathbb{N} \mid A \in \mathrm{Def}(M) \}$.
- Scott's problem: Are the standard systems exactly the Scott sets? (All standard systems are Scott sets and every Scott set of cardinality $\leq \aleph_1$ is a standard system.)
- Second-order arithmetic.

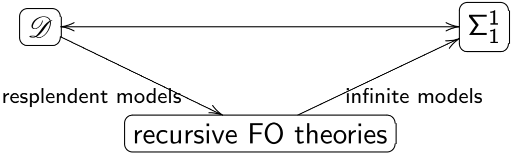
Recursive saturation

- A type over M is a set of formulas with finitely many parameters $\bar{a}$ from M and finitely many free variables $\bar{x}$ consistent with $\mathrm{Th}(M, \bar{a})$.
- M is recursively saturated if all recursive types over M are realized in M.
- PA: There is a Σ^1_1-sentence characterizing recursive saturation ("M-logic is consistent").
- Countable recursively saturated models of PA are nice: They are uniquely determined by its first-order theory and its standard system.

Resplendent models

- M is resplendent if for any recursive theory T in an expanded language $\mathscr{L} \supseteq \mathscr{L}_A \cup \{\bar{a}\}$ such that $T + \mathrm{Th}(M, \bar{a})$ is consistent there is an expansion of M satisfying T.
- All resplendent models are recursively saturated.
- All countable recursively saturated models are resplendent.
- PA: There is a Δ^1_2 sentence Θ characterizing resplendency.
- Θ says that M-logic is consistent and that for every sentence φ consistent in M-logic there is a satisfaction class including φ.

Dependence logic

- $=(x_1, \ldots, x_k, y)$ means $\exists f(f(x_1, \ldots, x_k) = y)$
- Why not "$x_1, \ldots, x_k \mapsto y$"?
- 

- Thus, there is a $\mathscr{D}$-sentence characterizing recursive saturation.
- Also, truth is definable in $\mathscr{D}$, i.e., there is a formula $\mathtt{Tr}(x)$ such that $M \models \varphi$ iff $M \models \mathtt{Tr}(\varphi)$ for models $M \models \mathrm{PA}$.

Logical constants in FOL I

- The semantic value of a formula $\varphi(\bar{x})$ on a model M is $\varphi(M|\bar{x}) = \{ \bar{a} \in M \mid M \models \varphi[\bar{a}/\bar{x}] \}$.
- Let $S_k(M) = \{ X \subseteq M^k \}$ be the set of all (possible) semantic values (of arity k). ($S_0 = \{ \mathtt{t}, \mathtt{f} \}$)
- A k-ary quantifier on M is a function $S_k \to \{ \mathtt{t}, \mathtt{f} \}$
- $\exists_M$ as a unary quantifier: $S_1(M) \to S_0$, $\exists_M(X) = \mathtt{t}$ iff $X \neq \emptyset$.
- A k-ary operator F on M is a set of functions $F^n : S_{n+k} \to S_n$.
- $\exists_M$ as a unary operator: $\exists^n_M : S_{n+1}(M) \to S_n(M)$ (projection).
- A k-ary quantifier Q gives rise to a k-ary operator:

$$S_{n+k}(M) \ni X \mapsto \left\{ \bar{b} \in M^n \;\middle|\; Q(X_{\bar{b}}) = \mathtt{t} \right\}$$

where $X_{\bar{b}} = \left\{ \bar{a} \in M^k \;\middle|\; \langle \bar{a}, \bar{b} \rangle \in X \right\}$ is the $\bar{b}$-slice of X.

Logical constants FOL II

- Which quantifiers are logical constants?
- Given a relation $\equiv$ on models, a quantifier Q respects $\equiv$ if given $(M, A) \equiv (N, B)$ we have $Q_M(A) = Q_N(B)$.
- Different relations $\equiv$ have been proposed for characterizing logical constants: Automorphic, isomorphic, homomorphic and back-and-forth equivalent.
- What if we shift view and work with operators instead of quantifiers? (Nothing happens in the isomorphism case.)

Logical constants in dependence logic I

- What is the semantic value of a formula in dependence logic?
- $T_k(M) = P(M^k) - \emptyset$
- $M \models_{T/\bar{x}} \varphi$
- $\lceil \bar{x} | \varphi \rceil_M = \{\, T \in T_k(M) \mid M \models_{T/\bar{x}} \varphi \,\}$
- $\lceil \varphi \rceil$ can not be the semantic value of φ if we want the semantics to be compositional wrt negation.
- $[\bar{x}|\varphi]$ be the *partial* function $T_k(M) \to \{\mathtt{t}, \mathtt{f}\}$ such that

$$[\bar{x}|\varphi]_M(T) = \begin{cases} \mathtt{t} & \text{if } M \models_{\bar{x}/T} \varphi \\ \mathtt{f} & \text{if } M \models_{\bar{x}/T} \neg\varphi \\ \text{undefined} & \text{otherwise} \end{cases}.$$

- Let $S_k(M)$ be the set of partial functions $T_k(M) \to \{\mathtt{t}, \mathtt{f}\}$.

Logical constants in dependence logic II

- A k-ary $\mathscr{D}$-quantifier on M is a partial function

$$Q_M : S_k(M) \to \{\, \mathtt{t}, \mathtt{f} \,\}.$$

- A team-structure is a set M together with $f \in S_k(M)$.
- Thus, a $\mathscr{D}$-quantifier Q is a partial function from the class of team-structures to $\{\, \mathtt{t}, \mathtt{f} \,\}$.
- Given a relation $\equiv$ on team structures a quantifier Q respects the relation if for any $(M, f) \equiv (N, g)$ we have $Q_M(f) = Q_N(g)$.
- To do: Characterize the quantifiers respecting certain (interesting) relations.

Quantifiers and $\mathscr{D}$-quantifiers

- Every quantifier Q gives, in a natural way, a $\mathscr{D}$-quantifier by

$$Q_M^{\mathscr{D}}(f) = \begin{cases} \mathtt{t} & \text{if } \cup f^{-1}(\mathtt{t}) \in Q_M \\ \mathtt{f} & \text{if } \cup f^{-1}(\mathtt{f}) \notin Q_M \\ \text{undefined} & \text{otherwise} \end{cases}$$

- If $\varphi(\bar{x}) \in \mathrm{FOL}$ and Q is a quantifier then $M \models Q\bar{x}\varphi(\bar{x})$ iff $M \models Q^{\mathscr{D}}\bar{x}\varphi(\bar{x})$.
- Test: Does this transformation respect logicality?

Thanks