Background

Transplendent models

Subtransplendent models

Further

Transplendent models

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Preliminaries

- All models will be (expansions of) models of PA.
- All languages $\mathcal{L}$ will be recursive extensions of the language of arithmetic, $\mathcal{L}_A$.
- The standard system of $M$, $\text{SSy}(M)$, is the collection of standard parts of (parameter) $\mathcal{L}_A$-definable sets in $M$.

$$\text{SSy}(M) = \{ X \cap \mathbb{N} \mid X \in \text{Def}(M') \} ,$$

where $M'$ is the $\mathcal{L}_A$-reduct of $M$. 
Recursive saturation

- A type over $M$ is a set of formulas with finitely many parameters $\bar{a}$ from $M$ and finitely many free variables $\bar{x}$ consistent with $\text{Th}(M, \bar{a})$.
- $M$ is recursively saturated if all recursive types over $M$ are realized in $M$.
- $M$ is recursively saturated iff all types in $\text{SSy}(M)$ are realized in $M$.
- There is a $\Sigma^1_1$-sentence $\Theta$ such that a model is recursively saturated iff it satisfies $\Theta$.
- $\Theta$ says that $M$-logic is consistent.
Resplendent models

- \( M \) is resplendent if for any theory \( T \) in an expanded language \( \mathcal{L} \supseteq \mathcal{L}_A \cup \{\bar{a}\} \) such that \( T + \text{Th}(M, \bar{a}) \) is consistent there is an expansion of \( M \) satisfying \( T \).
- All resplendent models are recursively saturated.
- All countable recursively saturated models are resplendent.
- There is a \( \Delta^1_2 \) sentence \( \Theta \) such that a model is resplendent iff it satisfies \( \Theta \).
- \( \Theta \) says that \( M \)-logic is consistent and that for every (non-standard) sentence \( \varphi \) consistent in \( M \)-logic there is a satisfaction class including \( \varphi \).
Subresplendent models

- $M$ is subresplendent if for any theory $T$ in an expanded language $\mathcal{L} \supseteq \mathcal{L}_A \cup \{\bar{a}\}$ such that $T + \text{Th}(M, \bar{a})$ is consistent there are an elementary submodel $\bar{a} \in N$ of $M$ and an expansion of $N$ satisfying $T$.

- A model is subresplendent iff it is recursively saturated.
Arithmetic saturation

- $M$ is arithmetically saturated if for any type arithmetic in some $\text{Th}(M, \bar{a})$, $\bar{a} \in M$, is realized in $M$.
- $M$ is arithmetically saturated iff $M$ is recursively saturated and $\text{SSy}(M)$ is closed under arithmetic comprehension.
- A countable recursively saturated model $M$ is arithmetically saturated iff there is a maximal automorphism, i.e. an automorphism $f$ such that

$$\text{fix}(f) = \{ a \in M \mid f(a) = a \} = \text{Scl}_M(\emptyset).$$
Omitting types

- $p^\uparrow$ is the $\mathcal{L}_{\omega_1\omega}$-sentence saying that $p$ is omitted, i.e.,

$$\forall \bar{x} \bigvee \neg \psi(\bar{x})$$

$$\psi(\bar{x}) \in p(\bar{x})$$

- $p(x)$ is isolated in $T$ if there is $\varphi(x)$ such that $T + \exists x \varphi(x)$ is consistent and $T \models \forall x (\varphi(x) \rightarrow \psi(x))$ for all $\psi(x) \in p(x)$.

- Omitting Types Theorem: If $p(x)$ is not isolated in $T$ then $T + p^\uparrow$ is consistent.
Omitting types, two examples

- If $M$ is countable and recursively saturated and $p(x)$ is a type which is not isolated in any $T + \text{Th}(M, \bar{a}, \bar{b})$ where $\bar{b} \in M$ (and these theories are consistent) then there is an expansion of $M$ satisfying $T + p \uparrow$.

- Maximal automorphism: Let $T_f$ say that $f$ is an automorphism and
  \[ p_f(x) = \{f(x) = x\} \cup \{x \neq t \mid t \text{ is a Skolem term}\}. \]
  $f$ is a maximal automorphism of $M$ iff $(M, f) \models T_f + p_f \uparrow$.
  $p_f$ is isolated in $\text{Th}(M, a)$, where $a \notin \text{Scl}_M(\emptyset)$.

- Standard cut: Let $T_K = \{K(n) \mid n \in \mathbb{N}\}$ and
  \[ p_K(x) = \{K(x)\} \cup \{x > n \mid n \in \mathbb{N}\}. \]
  $K$ is the standard cut of $M$ iff $(M, K) \models T_K + p_K \uparrow$.
  $p_K$ is isolated in $\text{Th}(M, a)$, where $a$ is non-standard.
Definition

- $T$ and $p(x)$ are a theory and a type in an extended language $\mathcal{L}$ with finitely many parameters $\bar{a}$ from $M$. $T_0$ a $\mathcal{L}_A$ theory.
- $T + p^\uparrow$ is fully consistent over $T_0$ if there is a model of $T_0 + T + p^\uparrow$ with standard system $\mathcal{P}(\mathbb{N})$ whose $\mathcal{L}_A$-reduct is recursively saturated.
- $T + p^\uparrow$ is fully consistent over $M$ if it is fully consistent over $\text{Th}(M, \bar{a})$.
- Let $M \models \text{PA}$. We say that $M$ is almost transplendent if for all $T, p(\bar{x}) \in \text{SSy}(M)$ such that $T + p^\uparrow$ is fully consistent over $M$, there is an expansion $M^+$ of $(M, \bar{a})$ such that $M^+ \models T + p^\uparrow$ and $\text{Th}(M^+, \bar{a}) + p^\uparrow$ is fully consistent over $M$. and $\text{Th}(M^+, \bar{a}) + p^\uparrow$ is fully consistent over $M$. 

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Transplendent models
Existence

Theorem

There is a transplendent model of cardinality $2^{\aleph_0}$.

Definition

A Scott set $\mathcal{K}$ is closed if for any $T_0, T, p \in \mathcal{K}$ such that $T + p \uparrow$ is fully consistent over $T_0$ there is a completion $T_c \in \mathcal{K}$ of $T$ such that $T_c + p \uparrow$ is fully consistent over $T_0$.

Theorem

Any countable recursively saturated model with a closed standard system is transplendent.
Closure is equivalent to transplendence

Given \( T, p \in \text{SSy}(M) \) such that \( T + p \uparrow \) is fully consistent over \( T_0 \) then the theory

\[
T + p \uparrow + \text{‘}c \text{ codes the theory of the model’}
\]

is fully consistent over \( T_0 \).

**Theorem**

Let \( M \) be a countable model of PA. \( M \) is transplendent iff \( M \) is recursively saturated and \( \text{SSy}(M) \) is closed.
The standard predicate

- Let $T_{K=\mathbb{N}}$ be the theory consisting of all $K(n), n \in \mathbb{N}$ together with $p^\uparrow$ where $p(x) = \{K(x)\} \cup \{x > n \mid n \in \mathbb{N}\}$.
- $(M, K) \models T_{K=\mathbb{N}}$ iff $K = \mathbb{N}$.
- Clearly $T_{K=\mathbb{N}}$ is fully consistent over any $T_0$. 
Second order arithmetic vs. the standard predicate

Given a second order $\mathcal{L}_A$-sentence $\Theta$ we translate it into a first order $\mathcal{L}_A(K)$-sentence $\Theta^K$:

- $(t \in X)^K$ becomes $(x)_t \neq 0$
- $(\exists x \psi)^K$ becomes $\exists x (K(x) \land \psi^K)$
- $(\exists X \psi)^K$ becomes $\exists x \psi^K$

**Theorem**

$(M, \mathbb{N}) \models \Theta^K(\bar{a})$ iff $SSy(M) \models \Theta(\bar{A})$, where $\bar{a}$ codes the sets $\bar{A}$.
Implications of transplendence

If $\mathcal{P}(\mathbb{N}) \models \Theta(\bar{A})$ then $T_{K=\mathbb{N}} + \Theta^K(\bar{a})$ is fully consistent over any $T_0$.

**Theorem**

*If M is transplendent then $SSy(M) \prec \mathcal{P}(\mathbb{N})$.*

Given $A \in SSy(M)$ then $T_{K=\mathbb{N}} + \text{`c codes } Th(\mathcal{P}(\mathbb{N}), A)\text{`} is fully consistent over any $T_0$.

**Theorem**

*If M is transplendent then $Th(\mathcal{P}(\mathbb{N}), A) \in SSy(M)$ for every $A \in SSy(M)$.*
Some basis theorems

- $\Gamma \subseteq \mathcal{P}(\mathbb{N})$ is a basis for $\Delta \subseteq \mathcal{P}(\mathcal{P}(\mathbb{N}))$ if $\forall X \in \Delta \ X \cap \Gamma \neq \emptyset$.
- $V = L$ implies that $\Delta^1_k$ is a basis for $\Sigma^1_k$ for all $k \geq 2$.
- $PD$ implies that $\Delta^1_k$ is a basis for $\Sigma^1_k$ for all even $k \geq 2$.
- If $V = L$ or $PD$ then $\Sigma^1_\infty$ is a basis for $\Sigma^1_\infty$.

**Theorem**

*If $V = L$ or $PD$ then if $\text{Th}(\mathcal{P}(\mathbb{N}), A) \in \mathcal{X}$ for every $A \in \mathcal{X}$ then $\mathcal{X} \prec \mathcal{P}(\mathbb{N})$.*

Is this true in general?
Weakening ‘fully consistent’

Replacing ‘fully consistent’ with ‘consistent’ in the definition of transplendent models will not work: Let $p(x)$ be some type realized in $M$.

**Theorem**

There are recursive $T, p$ such that

- For any type $q$ over PA there is a model of $PA + T + q\downarrow + p\uparrow$.
- No recursively saturated model of PA has an expansion satisfying $T + p\uparrow$.

$T + p\uparrow$ is $T_{K=\mathbb{N}} + \text{‘Σ is a truth predicate’} + \text{‘there is an omitted coded type’}$.
What can be done if ‘fully consistent’ is weakened to ‘consistent’?

**Definition**

We say that $M$ is *subtransplendent* if for all $T, p(\bar{x}) \in \text{SSy}(M)$ such that $T + p \uparrow + \text{Th}(M, \bar{a})$ is consistent there are an elementary submodel $(N, \bar{a}) \prec (M, \bar{a})$ and an expansion $N^+ \models T + p \uparrow$ of $(N, \bar{a})$.

**Theorem**

$M$ is subtransplendent iff $M$ is recursively saturated and for every $T, p \in \text{SSy}(M)$ such that $T + p \uparrow + \text{Th}(M, \bar{a})$ is consistent there is a completion $T_c \in \text{SSy}(M)$ of $T$ making $T_c + p \uparrow + \text{Th}(M, \bar{a})$ consistent.
**β-models**

**Definition**

\( \mathcal{X} \subseteq \mathcal{P}(\mathbb{N}) \) is a \( \beta \)-model if \( \mathcal{X} \prec \Sigma_1 \mathcal{P}(\mathbb{N}) \).

**Theorem**

\( M \) is subtransplendent iff \( M \) is recursively saturated and \( \text{SSy}(M) \) is a \( \beta \)-model.

Transplendence implies subtransplendence.
The logic of omitting a type

- A Scott set $\mathcal{X}$ is a $\beta$-model iff for every $T_0, T, p \in \mathcal{X}$ such that $T + p \uparrow + T_0$ is consistent there is a completion $T_c \in \mathcal{X}$ of $T$ making $T_c + p \uparrow + T_0$ consistent.

- For every hyperarithmetic $T$ and $p$ the height of a (minimal) proof in the logic of omitting a type (a variant on $\omega$-logic) is at most $\omega_1^{CK}$.

- There are hyperarithmetic $T$ and $p$ such that the supremum of the heights of (minimal) proofs is $\omega_1^{CK}$.

- The supremum of all heights of (minimal) proofs over all recursive $T, p$ is at least $\epsilon_0$. 
Open questions

- Is it possible to replace fully consistency with something weaker?
- Almost transplendence implies transplendence?
- Nicer characterization of the standard systems of transplendent models.
- What’s the complexity of the notion of transplendence/subtransplendence?
- Is there a satisfaction class type property such that $M$ is transplendent iff there is such a satisfaction class?