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Satisfaction classes
or
How to define truth in a non-standard world

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Abstract

In a non-standard model of PA there are non-standard sentences. Is it possible to define truth for these sentences? Tarski's theorem on the undefinability of truth says that there is no definable truth predicate, but we could still try to find an external (non-definable) predicate which is closed under Tarski's truth definition. These predicates are called *satisfaction classes*. It is not trivial that these predicates exist, in fact there are models of PA which do not admit them. Using an ingenious argument Lachlan proved that if a model admits a satisfaction class then it is recursively saturated. The converse also holds in the case of countable models, due to a theorem by Kotlarski, Krajewski and Lachlan. I will try to survey these results among others, and also present a new result telling us that the satisfaction classes can be made to respect non-standard propositional proofs. At the end I will make some remarks on future work in the area.

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I. Main definitions and su

The definition of satisfaction
important results in the area.

II. Proof sketches

Definition of $\mathfrak{M}$-logic and sket

III. New and future resu

A result concerning propositi
ideas on future work concerni

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Definition

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Introduction

$\mathfrak{M}$ is a non-standard model of $\mathtt{PA}$ in the language

$$\mathcal{L}_A = \{\mathsf{Succ}, +, \cdot, 0\}.$$

The logical symbols are $=, \neg, \vee, \exists$. The symbols $\wedge, \rightarrow, \leftrightarrow, \forall$ are abbreviations in the usual way.

Let

$$\mathcal{L}_{\mathfrak{M}} = \mathcal{L}_A \cup \{\mathsf{c}_a : a \in \mathfrak{M}\}.$$

Let ${}^*\mathcal{L}_{\mathfrak{M}}$ be the non-standard language corresponding to $\mathcal{L}_{\mathfrak{M}}$, i.e., the sentences of ${}^*\mathcal{L}_{\mathfrak{M}}$ are all $a \in \mathfrak{M}$ such that $\mathfrak{M} \models \mathtt{Sent}(a)$ where $\mathtt{Sent}$ is the formula binumerating the sentences in $\mathcal{L}_{\mathfrak{M}}$, i.e., for all $k \in \mathbb{N}$ $\mathtt{PA} \vdash \mathtt{Sent}(k)$ iff $k = \ulcorner\varphi\urcorner$ where φ is a ${}^*\mathcal{L}_{\mathfrak{M}}$ sentence.

Lemma 1 (Overspill). *If $\mathfrak{M} \models \varphi(k)$ for all $k \in \mathbb{N}$ then $\mathfrak{M} \models \varphi(a)$ for all $a < \nu$ for some non-standard ν.*

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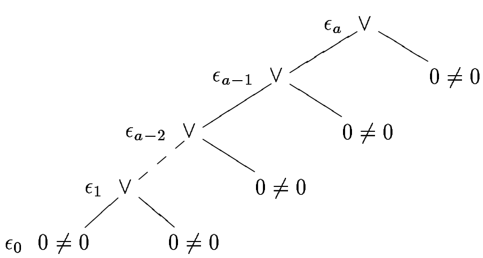
Example, non-standard sentence

$$\epsilon_0 \text{ is } 0 \neq 0$$
$$\epsilon_{a+1} \text{ is } \epsilon_a \vee 0 \neq 0$$



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Theorem 2 (Tarski's unde
formula $\mathtt{Tr}$ *such that*

$\mathfrak{M}$

for all standard sentences φ.
Gödel) code for φ.

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Satisf

Definition 3. A (full) satisfa
predicate on $\mathfrak{M}$ satisfying the

$\mathtt{Sent}(x)$

$\ulcorner t = r \urcorner \in \mathcal{S}$

$\ulcorner \neg\varphi \urcorner \in \mathcal{S}$

$\ulcorner \varphi \vee \psi \urcorner \in \mathcal{S}$

$\ulcorner \exists \mathsf{v}_i \gamma \urcorner \in \mathcal{S}$

for all ${}^*\mathcal{L}_{\mathfrak{M}}$-sentences φ, ψ and

First-order property so can be

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Examples

If $\mathfrak{M}$ is the standard model then there is exactly one satisfaction class

$$\mathcal{S}_0 = \{\ulcorner\varphi\urcorner : \mathfrak{M} \models \varphi\}.$$

If

$$(\mathbb{N}, \mathcal{S}_0) \prec (\mathfrak{M}, \mathcal{S})$$

then $\mathcal{S}$ is a satisfaction class on $\mathfrak{M}$.

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Recurs

Definition 4. A *recursive ty*

$$t(\bar{x}) =$$

of formulas with finitely many
parameters $\bar{a}$ such that the th
$t(\bar{x})$ is *realized* in $\mathfrak{M}$ if there a
for all $\varphi(\bar{x}) \in t(\bar{x})$.

Definition 5. $\mathfrak{M}$ is *recursive*
are realized.

Proposition 6. *If $\mathfrak{M}$ is any*
there is an elementary extens
and such that $|\mathfrak{N}| = |\mathfrak{M}|$.

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Historical outline

Robinson 1963: External and internal truth.

Krajewski 1976: Defines and investigates satisfaction classes.

Kotlarski, Krajewski, Lachlan 1981: Proves existence theorems.

Smith 1984: Proves some strengthenings of the KKL results and obtain characterizations of recursive saturation and resplendency.

Other important names: Ratajczyk, Kossak, Murawski.

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Res

Definition 7. $\mathfrak{M}$ is *resplend*
that $\mathtt{Th}(\mathfrak{M}) \cup \{\Phi\}$ is consister

Theorem 8 ([Kle52]). *If $\mathfrak{M}$*
saturated.

There is a converse if the mod
Barwise and Schlipf and indep

Theorem 9 ([BS76]). *If $\mathfrak{M}$*
then it is resplendent.

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Main theorems

Theorem 10 ([KKL81]). *If $\mathfrak{M}$ is a countable recursively saturated model of* `PA` *then it admits a satisfaction class.*

Theorem 11 ([Lac81]). *If $\mathfrak{M}$ (non-standard model of* `PA`*) admits a satisfaction class then it is recursively saturated.*

Theorem 12 ([Smi84]). *There is a Σ^1_1 formula Φ such that for any model $\mathfrak{M}$ of* `PA`

$$\mathfrak{M} \models \Phi \text{ iff } \mathfrak{M} \text{ is recursively saturated,}$$

i.e., recursive saturation is a Σ^1_1 property.

Theorem 13 ([Smi84]). *Resplendency is a Δ^1_2 property.*

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Proof sketches

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The

Theorem 14. *If $\mathfrak{M}$ is a recu*
and $a \in \mathfrak{M} - \mathbb{N}$ then there is
ϵ_a true.

Proof plan.

1. Define $\mathfrak{M}$-logic.
2. Prove consistency of $\mathfrak{M}$-lo
3. Prove a completeness the

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$\mathfrak{M}$-logic is a formal deduction
$^*\mathcal{L}_{\mathfrak{M}}$-sentences. Similar to ω –
$\Gamma \cup \{\varphi\}$.

$\varphi, \neg\varphi$

$t = r$ if $\mathfrak{M}$

$t \neq r$ if $\mathfrak{M}$

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$$\frac{\Gamma}{\Gamma, \varphi} \qquad \text{(Weakening)}$$

$$\frac{\Gamma, \varphi}{\Gamma, \varphi \vee \psi} \qquad \text{(I}\vee 1)$$

$$\frac{\Gamma, \psi}{\Gamma, \varphi \vee \psi} \qquad \text{(I}\vee 2)$$

$$\frac{\Gamma, \neg\varphi \quad \Gamma, \neg\psi}{\Gamma, \neg(\varphi \vee \psi)} \qquad \text{(I}\vee 3)$$

$$\frac{\Gamma, \varphi}{\Gamma, \neg\neg\varphi} \qquad \text{(I}\neg)$$

$$\frac{\Gamma, \varphi \quad \Gamma, \neg\varphi}{\Gamma} \qquad \text{(Cut)}$$

$$\frac{\Gamma, \varphi[\mathsf{c}_a/\mathsf{v}_i]}{\Gamma, \exists \mathsf{v}_i \varphi} \qquad \text{(I}\exists)$$

$$\frac{\Gamma, \neg\varphi[\mathsf{c}_a/v_i] \text{ for all } a \in \mathfrak{M}}{\Gamma, \neg\exists v_i \varphi} \qquad (\mathfrak{M}\text{-rule})$$

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Consistency of $\mathfrak{M}$-logic

Let $\vdash_{\mathfrak{M}}$ denote provability in $\mathfrak{M}$-logic and $\vdash_{\mathfrak{M}}^{\omega}$ provability in $\mathfrak{M}$-logic with proofs of finite height.

Lemma 15. *If $\mathfrak{M}$ is recursively saturated then $\mathfrak{M}$-logic is finite, i.e., if $\vdash_{\mathfrak{M}} \Gamma$ then $\vdash_{\mathfrak{M}}^{\omega} \Gamma$.*

By looking only at finite depths of formulas we can *approximate* them by standard formulas (by adding extra predicate symbols). Let $\vdash_{\mathrm{T}}$ denote provability of these approximations in first-order logic with the $\mathfrak{M}$-rule.

Lemma 16. *If $\vdash_{\mathfrak{M}}^{\omega} \Gamma$ then there is an approximation Δ of Γ such that $\vdash_{\mathrm{T}} \Delta$.*

The consistency of $\mathfrak{M}$-logic follows since all approximations of $0 \neq 0$ is $0 \neq 0$ and $\nvdash_{\mathrm{T}} 0 \neq 0$.

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Construction

Lemma 17. *If $\mathfrak{M}$ is countab*
there is a satisfaction class e

$\varphi_1, \varphi_2, \ldots$ enumeration of all

$$\Gamma_0 = \Gamma$$

$$\Gamma_{i+1} = \begin{cases} \Gamma_i, \exists x\gamma, \gamma(\mathsf{c}_a) \\ \Gamma_i, \varphi_{i+1} \\ \Gamma_i, \neg\varphi_{i+1} \end{cases}$$

where a is such that $\Gamma_i \nvdash_{\mathfrak{M}} \neg\gamma$

is a satisfaction class.

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Lachlan

$\mathcal{S}$ inductive satisfaction class
$\mathcal{L}_A \cup \{\mathcal{S}\}$.

$\mathcal{S}$ a inductive satisfaction clas
extends to nonstandard i by

$$\mathfrak{M} \vDash \exists$$

for some nonstandard a. The
recursively saturated.

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Lachlan's result, proof

$\{\varphi_i\}$ non-realized recursive type.
$A_i = \{x \in \mathfrak{M} : \mathfrak{M} \models \varphi_i(x)\}$.
Can assume $A_{i+1} \subsetneq A_i$ and $A_0 = \mathfrak{M}$.
$B_0 = \emptyset, B_{i+1} = A_i - A_{i+1}$, $\{B_i\}_{i=1}^{\infty}$ partition of $\mathfrak{M}$.

$$C_0 = \emptyset$$

$$C_{i+1} = \begin{cases} B_1 & \text{if } C_i = \emptyset, \\ B_{j+1} & \text{if } j = (\mu j) B_j \cap C_i \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

Define these sets for non-standard $i < \nu$ by using a satisfaction class (they are all first order properties).

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- $\forall i < \nu \exists j \in \mathbb{N} C_i = B_j$
- $C_i = B_j \to C_{i+1} = B_{j+1}$
- $C_i \neq \emptyset$ if $i > 0$.

$$f :< \nu \to \mathbb{N}, i \mapsto (\mu j) C_i = B_j$$

is total and $f(i+1) = f(i) + 1$, so

$$f(\nu - 1) > f(\nu - 2) > f(\nu - 3) > \ldots$$

infinite descending sequence of natural numbers.
CONTRADICTION.

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New

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Intro

$\Sigma_k - \texttt{PA}(\mathcal{S})$ is the theory

$$\texttt{PA} + \texttt{SatCl}(\mathcal{S}) +$$

Theorem 18 ([Kot85]). *Th*
arithmetical part of

$$\texttt{PA} + \texttt{SatCl}(\mathcal{S}$$

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Propositional proofs

Definition 19. A satisfaction class $\mathcal{S}$ is said to be *closed under propositional logic* if

$$\mathfrak{M} \models \mathcal{S} \vdash_{\text{prop}} \varphi \text{ implies that } \varphi \in \mathcal{S},$$

where $\mathcal{S} \vdash_{\text{prop}} \varphi$ is the arithmetical formula saying that φ is provable from $\mathcal{S}$ in propositional logic.

Theorem 20. *If $\mathfrak{M}$ is recursively saturated then it admits a satisfaction class closed under propositional logic.*

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Consisten

Consistency proof by constru

then $\mathfrak{M} \models \mathtt{Tr}_i(\varphi)$.

Consistency then follows from

Remark. The consistency of

simpler argument then KKL,

$\mathfrak{M}$-logic (when $\mathfrak{M}$ is rec. sat.)

The satisfaction class is const

the important exception that

φ_i. This makes the satisfactio

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$\mathfrak{M}_{\text{prop}}$-logic

$$\frac{\Lambda}{\Gamma} \quad \text{if } \mathfrak{M} \models \bigvee\!\!\!\bigvee \Lambda \vdash_{\text{prop}} \bigvee\!\!\!\bigvee \Gamma. \qquad \text{(Prop)}$$

$$\frac{\Gamma, \varphi[\mathsf{c}_a/\mathsf{v}_i]}{\Gamma, \exists \mathsf{v}_i \varphi} \qquad \text{(I}\exists\text{)}$$

$$\frac{\Gamma, \neg\varphi[\mathsf{c}_a/v_i] \text{ for all } a \in \mathfrak{M}}{\Gamma, \neg\exists v_i \varphi} \qquad (\mathfrak{M}\text{-rule})$$

Provability is denoted by $\vdash_{\mathfrak{M}_{\text{prop}}}$. If $\mathfrak{M}$ is recursively saturated then $\mathfrak{M}_{\text{prop}}$-logic is finite.

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Fu

Definition 21. $\mathfrak{M}$ is **arithm**

saturated and for all definable

$c \in \mathfrak{M}$ such that

$$\forall n \in \mathbb{N}(f($$

Theorem 22 ([KKK91]). *I*

arithmetically saturated iff th

that $fix(g) = \mathfrak{M}_0$, where $fix(g)$

definable points in $\mathfrak{M}$.

Question 23. Is there a way o

saturation in terms of satisfac

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