

# Logical constants: Invariance and definability

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- ③ General invariance
- ④ Borel quantifiers

- Everyone living in Djursholm is wealthy. I live in Djursholm. Therefore I'm wealthy.
- Everyone living in **Botkyrka** is wealthy. I live in **Botkyrka**. Therefore I'm wealthy.
- Everyone living in Djursholm is wealthy. **Björn** lives in Djursholm. Therefore **Björn** is wealthy.
- **Someone** living in Djursholm is wealthy. I live in Djursholm. Therefore I'm wealthy.

# An “inferential” approach

$$\forall x(Px \rightarrow Rx)$$

$$\frac{Pc}{Rc}$$

$$\forall x(Px \rightarrow Qx)$$

$$\frac{Pc}{Qc}$$

$$\forall x(Px \rightarrow Rx)$$

$$\frac{Pd}{Rd}$$

$$\forall x(Px \vee Rx)$$

$$\frac{Pc}{Rc}$$

## A “model theoretic” approach

*An operator (function/predicate) is a logical constant if it is **topic neutral**.*

- Examples:  $\exists$ ,  $\forall$ ,  $\neg$ , and  $\rightarrow$ .
- Non-example: “for all even numbers”
- Debatable: “for infinitely many”, =

Mautner, Tarski, Mostowski, Lindenbaum: Logic is the the study of the invariants under the most general transformations (=permutations). (Klein’s Erlangen program)

## Definition (Lindström/Mostowski)

A (global) **generalized quantifier**  $Q$  of type  $\langle n_1, \dots, n_k \rangle$  is a (class) of structures in the language  $\{ R_1, \dots, R_k \}$  where  $R_i$  is of arity  $n_i$ .

Examples:

- $\exists = \{ (M, A) \mid A \subseteq M, A \neq \emptyset \}$
- $\forall = \{ (M, M) \mid M \}$
- $Q_0 = \{ (M, A) \mid A \subseteq M, |A| \geq \aleph_0 \}$
- $\exists^{\kappa} = \{ (M, A) \mid A \subseteq M, |A| = \kappa \}$
- $I = \{ (M, A, B) \mid A, B \subseteq M, |A| = |B| \}$
- $W = \{ (M, R) \mid R \subseteq M^2, R \text{ is well-founded} \}$
- $Q^A = \{ (M, B) \mid A \subseteq B \}$

- $\varphi(M) = \{ \bar{a} \in M^k \mid M \models \varphi(\bar{a}) \}$
- $M \models Qx_0 \dots x_{k-1} \varphi(x_0, \dots, x_{k-1})$  iff  $(M, \varphi(M)) \in Q$  ( $Q$  of type  $\langle k \rangle$ )

Local versions: For a given domain  $M$ , let (for  $Q$  of type  $\langle k \rangle$ )

$$Q_M = \left\{ R \subseteq M^k \mid (M, R) \in Q \right\}.$$

A (local) quantifier  $Q_M$  of type  $\langle k \rangle$  is definable in the logic  $\mathcal{L}$  if there is  $\varphi$  of  $\mathcal{L}$ , such that

$$(M, R) \models \varphi \text{ iff } R \in Q_M.$$

## Tarski's thesis

A (local) quantifier on a domain  $M$  is a logical constant iff it is invariant under all **permutations** of  $M$ .

Examples:  $\exists, \forall, Q_0, \exists^{\neq \kappa}, I$

Non-examples:  $Q^A$

## Mostowski's thesis

A quantifier  $Q$  is a logical constant iff it is invariant under all **bijections** (across domains).

## Theorem (McGee -91 / Krasner -38)

$Q$  is bijection invariant iff for each  $\kappa$  there is a formula in  $\mathcal{L}_{\infty\infty}$  defining  $Q_\kappa$ .

Fix a domain  $\Omega$ . Quantifier means local quantifier on  $\Omega$ .

$\mathcal{Q}$  is a set of quantifiers.

$G$  subgroup of the full symmetric group  $S_\Omega$ .

## Definition

- Let  $\text{Aut}(\mathcal{Q})$  be the group of all permutations of  $\Omega$  fixing all quantifiers in  $\mathcal{Q}$ :

$$\text{Aut}(\mathcal{Q}) = \{ g \in S_\Omega \mid g(Q) = Q \text{ for all } Q \in \mathcal{Q} \}.$$

- Let  $\text{Inv}(G)$  be the set of quantifiers fixed by  $G$ :

$$\text{Inv}(G) = \{ Q \mid g(Q) = Q \text{ for all } g \in G \}.$$

## Theorem (Krasner/Bonnay/E)

- $\text{Aut}(\text{Inv}(G)) = G$
- $\text{Inv}(\text{Aut}(\mathcal{Q}))$  is the set of quantifiers definable in  $\mathcal{L}_{\infty\infty}(\mathcal{Q})$

## Proof

**$\text{Aut}(\text{Inv}(G)) = G$** : Let  $\leq$  well-order  $\Omega$ , and  $Q = \{g(\leq) \mid g \in G\}$  of type  $\langle 2 \rangle$ . If  $h \in \text{Aut}(\text{Inv}(G))$  then  $h(\leq) \in Q$  and so there is  $g \in G$  such that  $h(\leq) = g(\leq)$ , implying  $h = g$ .

**$\text{Inv}(\text{Aut}(\mathcal{Q}))$  is the set of  $Q$ s definable in  $\mathcal{L}_{\infty\infty}(\mathcal{Q})$** : We assume all quantifiers of type  $\langle 1 \rangle$  and  $\Omega = \omega$ .  $Q' \in \text{Inv}(\text{Aut}(\mathcal{Q}))$  is defined by

$$\forall x_0, x_1, \dots \left[ \bigwedge_{i \neq j} x_i \neq x_j \wedge \forall y \bigvee_i y = x_i \wedge \bigwedge_{Q \in \mathcal{Q}} \left( \left( \bigwedge_{A \in Q} Qy \bigvee_{i \in A} y = x_i \right) \wedge \left( \bigwedge_{A \notin Q} \neg Qy \bigvee_{i \in A} y = x_i \right) \right) \rightarrow \bigvee_{A \in Q'} \left( \bigwedge_{i \in A} P_{x_i} \wedge \bigwedge_{i \notin A} \neg P_{x_i} \right) \right]$$

## Theorem

If  $\text{Inv}_m(G)$  are all **monadic** quantifiers invariant under  $G$  then there is a subgroup  $G$  such that  $\text{Aut}(\text{Inv}_m(G)) \supsetneq G$ .

Proof. Let  $G$  be the group of **piecewise monotone** permutations on  $\omega$ :  $g \in S_\omega$  is piecewise monotone if there exists partitions  $A_1 \cup \dots \cup A_k = B_1 \cup \dots \cup B_k = \omega$  such that  $g|_{A_i}$  is the unique increasing function  $A_i \rightarrow B_i$ .

$\text{Aut}(\text{Inv}_m(G))$  is closed in the topology generated by

$$U_{\bar{A}, \bar{B}} = \{ h \in S_\omega \mid h(A_i) = B_i \text{ all } i < k \}$$

as basic open sets, where  $\bar{A} = A_0, \dots, A_{k-1}$  and  $\bar{B} = B_0, \dots, B_{k-1}$  are subsets of  $\omega$ .

The closure of  $G$  is  $S_\omega$ .

# Feferman's thesis -99

## Definition

A (global) quantifier  $Q$  is invariant under **preimages of surjections** if for every  $h : M \rightarrow N$  surjection and for all  $R \subseteq N^k$ :  $h^{-1}(R) \in Q_M$  iff  $R \in Q_N$ .

## Theorem (Feferman)

Quantifiers of type  $\langle 1, \dots, 1 \rangle$  are invariant under **preimages of surjections** iff they are definable in  $\mathcal{L}_{\omega\omega}^-$ .

## Feferman's thesis

A quantifier is a logical constant iff it can be defined (in typed  $\lambda$ -calculus) from equality and monadic quantifiers invariant under preimages of surjections.

$h : M \rightarrow N$  can be “lifted” by:  $h(Q_M) = \{ h(R) \mid R \in Q_M \}$ .  
 Invariance under **surjections**:  $h(Q_M) = Q_N$  for all surjective  $h$ .

### Theorem (Casanovas -07)

*“Quantifiers” are invariant under surjections iff they are definable in a certain positive fragment of  $\mathcal{L}_{\omega\omega}$  (with restricted use of equality).*

Invariance under **back-and-forth equivalence**: If  $(M, A)$  and  $(N, B)$  are back-and-forth equivalent, then  $A \in Q_M$  iff  $B \in Q_N$ .

### Theorem (Barwise -73)

A local quantifier  $Q$  on  $M$  is **back-and-forth invariant** iff  $Q$  is definable in  $\mathcal{L}_{\infty\omega}$ .

Bonnay (BSL -08) argues well for that **if** the logical constants are **the invariants** under some relation between structures, then this relation is **back-and-forth equivalence**.

Assume now all quantifiers are local quantifiers on  $\omega$ .

### Theorem (Lopez-Escobar)

*A quantifier is **Borel** and **permutation invariant** iff it is definable in  $\mathcal{L}_{\omega_1\omega}$ .*

Indicates a strong connection between  $\mathcal{L}_{\omega_1\omega}$  and Borel quantifiers.

### FALSE

*$Q$  is Borel and  $\text{Aut}(\mathcal{Q})$  invariant iff  $Q$  is definable in  $\mathcal{L}_{\omega_1\omega}(\mathcal{Q})$ .*

Let  $A \subseteq \omega$  be infinite and coinfinite and  $Q' = \{A\}$  then  $Q^A$  is  $\text{Aut}(Q')$  invariant, but not definable in  $\mathcal{L}_{\omega_1\omega}(Q')$ .

## Theorem (E/Schlicht)

*Let  $\mathcal{Q}$  be a countable set of clopen quantifiers. Then  $Q$  is Borel and  $\text{Aut}(\mathcal{Q})$  invariant iff it is definable in  $\mathcal{L}_{\omega_1\omega}(\mathcal{Q})$ .*

## Question

*For which sets  $\mathcal{Q}$  of quantifiers does the theorem hold?*

# Thanks