Logical constants: Invariance and definability

Fredrik Engström
Institut Mittag-Leffler

2009–10–22
1. Introduction

2. Permutation invariance

3. General invariance

4. Borel quantifiers
• Everyone living in Djursholm is wealthy. I live in Djursholm. Therefore I’m wealthy.

• Everyone living in Botkyrka is wealthy. I live in Botkyrka. Therefore I’m wealthy.

• Everyone living in Djursholm is wealthy. Björn lives in Djursholm. Therefore Björn is wealthy.

• Someone living in Djursholm is wealthy. I live in Djursholm. Therefore I’m wealthy.
An “inferential” approach

∀x(Px → Rx)

\[ \frac{Pc}{Rc} \]

∀x(Px → Qx)

\[ \frac{Pc}{Qc} \]

∀x(Px → Rx)

\[ \frac{Pd}{Rd} \]

∀x(Px ∨ Rx)

\[ \frac{Pc}{Rc} \]

∃x (Px → Rx)
A “model theoretic” approach

An operator (function/predicate) is a logical constant if it is topic neutral.

- Examples: \( \exists \), \( \forall \), \( \neg \), and \( \rightarrow \).
- Non-example: “for all even numbers”
- Debatable: “for infinitely many”, =

Mautner, Tarski, Mostowski, Lindenbaum: Logic is the study of the invariants under the most general transformations (\(=\)permutations). (Klein’s Erlangen program)
Definition (Lindström/Mostowski)

A (global) **generalized quantifier** $Q$ of type $\langle n_1, \ldots, n_k \rangle$ is a (class) of structures in the language $\{ R_1, \ldots, R_k \}$ where $R_i$ is of arity $n_i$.

Examples:

- $\exists = \{ (M, A) \mid A \subseteq M, A \neq \emptyset \}$
- $\forall = \{ (M, M) \mid M \}$
- $Q_0 = \{ (M, A) \mid A \subseteq M, |A| \geq \aleph_0 \}$
- $\exists^=\kappa = \{ (M, A) \mid A \subseteq M, |A| = \kappa \}$
- $I = \{ (M, A, B) \mid A, B \subseteq M, |A| = |B| \}$
- $W = \{ (M, R) \mid R \subseteq M^2, R \text{ is well-founded} \}$
- $Q^A = \{ (M, B) \mid A \subseteq B \}$
$\varphi(M) = \{ \bar{a} \in M^k \mid M \models \varphi(\bar{a}) \}$

- $M \models Qx_0 \ldots x_{k-1} \varphi(x_0, \ldots, x_{k-1})$ iff $(M, \varphi(M)) \in Q$ (Q of type $\langle k \rangle$)

Local versions: For a given domain $M$, let (for $Q$ of type $(\langle k \rangle)$

$$Q_M = \left\{ R \subseteq M^k \mid (M, A) \in Q \right\}.$$ 

A (local) quantifier $Q_M$ of type $\langle k \rangle$ is definable in the logic $\mathcal{L}$ if there is $\varphi$ of $\mathcal{L}$, such that

$$(M, R) \models \varphi \text{ iff } R \in Q_M.$$
Tarski’s thesis

A (local) quantifier on a domain $M$ is a logical constant iff it is invariant under all permutations of $M$.

Examples: $\exists, \forall, Q_0, \exists^=\kappa, I$
Non-examples: $Q^A$

Mostowski’s thesis

A quantifier $Q$ is a logical constant iff it is invariant under all bijections (across domains).

Theorem (McGee -91 / Krasner -38)

$Q$ is bijection invariant iff for each $\kappa$ there is a formula in $\mathcal{L}_{\infty\infty}$ defining $Q_\kappa$.
Fix a domain $\Omega$. Quantifier means local quantifier on $\Omega$.
$\mathcal{Q}$ is a set of quantifiers.
$G$ subgroup of the full symmetric group $S_\Omega$.

**Definition**

- Let $\text{Aut}(\mathcal{Q})$ be the group of all permutations of $\Omega$ fixing all quantifiers in $\mathcal{Q}$:
  \[ \text{Aut}(\mathcal{Q}) = \{ g \in S_\Omega \mid g(Q) = Q \text{ for all } Q \in \mathcal{Q} \} . \]
- Let $\text{Inv}(G)$ be the set of quantifiers fixed by $G$:
  \[ \text{Inv}(G) = \{ Q \mid g(Q) = Q \text{ for all } g \in G \} . \]

**Theorem (Krasner/Bonnay/E)**

- $\text{Aut}(\text{Inv}(G)) = G$
- $\text{Inv}(\text{Aut}(\mathcal{Q}))$ is the set of quantifiers definable in $L_\infty(\mathcal{Q})$.
Proof

\textbf{Aut(Inv(}G) \textbf{)} = G: Let \( \leq \) well-order \( \Omega \), and \( Q = \{ g(\leq) \mid g \in G \} \) of type \( \langle 2 \rangle \). If \( h \in \text{Aut(Inv(}G) \text{)} \) then \( h(\leq) \in Q \) and so there is \( g \in G \) such that \( h(\leq) = g(\leq) \), implying \( h = g \).

\textbf{Inv(Aut(}Q) \textbf{)) is the set of Qs definable in } L_{\infty\infty}(Q): \textbf{ We assume all quantifiers of type } \langle 1 \rangle \textbf{ and } \Omega = \omega . \textbf{ Q'} \in \text{Inv(Aut(}Q) \text{))} \textbf{ is defined by}

\[ \forall x_0, x_1, \ldots \left[ \bigwedge_{i \neq j} x_i \neq x_j \land \forall y \bigvee_i y = x_i \right] \land \bigwedge_{Q \in \mathcal{Q}} \left( \bigwedge_{A \in Q} \bigvee_{i \in A} y = x_i \right) \land \left( \bigwedge_{A \notin Q} \bigvee_{i \in A} y = x_i \right) \rightarrow \bigvee_{A \in Q'} \left( \bigwedge_{i \in A} P_{x_i} \land \bigwedge_{i \notin A} \neg P_{x_i} \right) \]
Theorem

If $\text{Inv}_m(G)$ are all **monadic** quantifiers invariant under $G$ then there is a subgroup $G$ such that $\text{Aut}(\text{Inv}_m(G)) \supseteq G$.

Proof. Let $G$ be the group of **piecewise monotone** permutations on $\omega$: $g \in S_\omega$ is piecewise monotone if there exists partitions $A_1 \cup \ldots \cup A_k = B_1 \cup \ldots \cup B_k = \omega$ such that $g|A_i$ is the unique increasing function $A_i \rightarrow B_i$.

$\text{Aut}(\text{Inv}_m(G))$ is closed in the topology generated by

$$U_{\bar{A},\bar{B}} = \{ h \in S_\omega \mid h(A_i) = B_i \text{ all } i < k \}$$

as basic open sets, where $\bar{A} = A_0, \ldots, A_{k-1}$ and $\bar{B} = B_0, \ldots, B_{k-1}$ are subsets of $\omega$.

The closure of $G$ is $S_\omega$. 
### Definition

A (global) quantifier $Q$ is invariant under **preimages of surjections** if for every $h : M \to N$ surjection and for all $R \subseteq N^k$: 

$$h^{-1}(R) \in Q_M \iff R \in Q_N.$$  

### Theorem (Feferman)

Quantifiers of type $\langle 1, \ldots, 1 \rangle$ are invariant under **preimages of surjections** iff they are definable in $\mathcal{L}_{\omega\omega}$.

### Feferman’s thesis

A quantifier is a logical constant iff it can be defined (in typed $\lambda$-calculus) from equality and monadic quantifiers invariant under preimages of surjections.
$h : M \to N$ can be “lifted” by: $h(Q_M) = \{ h(R) \mid R \in Q_M \}$.

Invariance under **surjections**: $h(Q_M) = Q_N$ for all surjective $h$.

**Theorem (Casanovas -07)**

“Quantifiers” are invariant under surjections iff they are definable in a certain positive fragment of $L_{\omega \omega}$ (with restricted use of equality).

Invariance under **back-and-forth equivalence**: If $(M, A)$ and $(N, B)$ are back-and-forth equivalent, then $A \in Q_M$ iff $B \in Q_N$.

**Theorem (Barwise -73)**

A local quantifier $Q$ on $M$ is **back-and-forth invariant** iff $Q$ is definable in $L_{\infty \omega}$.

Bonnay (BSL -08) argues well for that **if** the logical constants are the invariants under some relation between structures, then this relation is **back-and-forth equivalence**.
Assume now all quantifiers are local quantifiers on $\omega$.

**Theorem (Lopez-Escobar)**

A quantifier is **Borel and permutation invariant** iff it is definable in $L_{\omega_1\omega}$.

Indicates a strong connection between $L_{\omega_1\omega}$ and Borel quantifiers.

**FALSE**

$Q$ is Borel and $\text{Aut}(Q)$ invariant iff $Q$ is definable in $L_{\omega_1\omega}(Q)$.

Let $A \subseteq \omega$ be infinite and coinfinite and $Q' = \{ A \}$ then $Q^A$ is $\text{Aut}(Q')$ invariant, but not definable in $L_{\omega_1\omega}(Q')$. 

Logical constants: Invariance and definability

Fredrik Engström Institut Mittag-Leffler
Theorem (E/Schlicht)

Let $\mathcal{Q}$ be a countable set of clopen quantifiers. Then $\mathcal{Q}$ is Borel and $\text{Aut}(\mathcal{Q})$ invariant iff it is definable in $\mathcal{L}_{\omega_1\omega}(\mathcal{Q})$.

Question

For which sets $\mathcal{Q}$ of quantifiers does the theorem hold?
Thanks