Borel quantifiers

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Logical constants

An operator (function/predicate) is a logical constant if it is **topic neutral**.

- Examples: $\exists$, $\forall$, $\neg$, and $\rightarrow$.
- Non-example: "for all even numbers"
- Debatable: "for infinitely many", $=$

Mautner, Tarski, Mostowski, Lindenbaum: Logic is the the study of the invariants under the most general transformations (=permutations). (Klein's Erlangen program)

Definition (Lindström/Mostowski)

A (global) **generalized quantifier** *Q of type $\langle n_1, \ldots, n_k \rangle$ is a (class) of structures in the language $\{ R_1, \ldots, R_k \}$ where R_i is of arity n_i.*

Examples:

- $\exists = \{ (M, A) \mid A \subseteq M, A \neq \emptyset \}$
- $\forall = \{ (M, M) \mid M \}$
- $Q_0 = \{ (M, A) \mid A \subseteq M, |A| \geq \aleph_0 \}$
- $\exists^{=\kappa} = \{ (M, A) \mid A \subseteq M, |A| = \kappa \}$
- $I = \{ (M, A, B) \mid A, B \subseteq M, |A| = |B| \}$
- $W = \{ (M, R) \mid R \subseteq M^2, R \text{ is well-founded} \}$
- $Q^A = \{ (M, B) \mid A \subseteq B \}$

- $\varphi(M) = \{ \bar{a} \in M^k \mid M \models \varphi(\bar{a}) \}$
- $M \models Qx_0 \ldots x_{k-1}\varphi(x_0, \ldots, x_{k-1})$ *iff* $(M, \varphi(M)) \in Q$ *(Q of type $\langle k \rangle$)*

Local versions: For a given domain M, let (for Q of type $(\langle k \rangle)$

$$Q_M = \left\{ R \subseteq M^k \,\middle|\, (M, A) \in Q \right\}.$$

A (local) quantifier Q_M of type $\langle k \rangle$ is definable in the logic $\mathscr{L}$ if there is φ of $\mathscr{L}$, such that

$$(M, R) \models \varphi \text{ iff } R \in Q_M.$$

Tarski's thesis

A (local) quantifier on a domain M is a logical constant iff it is invariant under all **permutations** *of M.*

Examples: $\exists, \forall, Q_0, \exists^{=\kappa}, I$
Non-examples: Q^A

Mostowski's thesis

A quantifier Q is a logical constant iff it is invariant under all **bijections** *(across domains).*

Theorem (McGee -91 / Krasner -38)

Q is bijection invariant iff for each κ there is a formula in $\mathscr{L}_{\infty\infty}$ defining Q_κ.

Fix a domain Ω. Quantifier means local quantifier on Ω.
$\mathscr{Q}$ is a set of quantifiers.
G subgroup of the full symmetric group S_Ω.

Definition

- *Let* Aut($\mathscr{Q}$) *be the group of all permutations of* Ω *fixing all quantifiers in* $\mathscr{Q}$*:*
  $\mathrm{Aut}(\mathscr{Q}) = \{\, g \in S_\Omega \mid g(Q) = Q \text{ for all } Q \in \mathscr{Q} \,\}$.
- *Let* Inv(G) *be the set of quantifiers fixed by* G*:*
  $\mathrm{Inv}(G) = \{\, Q \mid g(Q) = Q \text{ for all } g \in G \,\}$.

Theorem (Krasner/Bonnay/E)

- $\mathrm{Aut}(\mathrm{Inv}(G)) = G$
- $\mathrm{Inv}(\mathrm{Aut}(\mathscr{Q}))$ *is the set of quantifiers definable in* $\mathscr{L}_{\infty\infty}(\mathscr{Q})$

Proof

$\mathrm{Aut}(\mathrm{Inv}(G)) = G$: Let $\leq$ well-order Ω, and $Q = \{ g(\leq) \mid g \in G \}$ of type $\langle 2 \rangle$. If $h \in \mathrm{Aut}(\mathrm{Inv}(G))$ then $h(\leq) \in Q$ and so there is $g \in G$ such that $h(\leq) = g(\leq)$, implying $h = g$.

$\mathrm{Inv}(\mathrm{Aut}(\mathscr{Q}))$ is the set of Qs definable in $\mathscr{L}_{\infty\infty}(\mathscr{Q})$: We assume all quantifiers of type $\langle 1 \rangle$ and $\Omega = \omega$.
$Q' \in \mathrm{Inv}(\mathrm{Aut}(\mathscr{Q}))$ is defined by

$$\forall x_0, x_1, \ldots \Big[\bigwedge_{i \neq j} x_i {\neq} x_j \wedge \forall y \bigvee_i y{=}x_i \wedge \bigwedge_{Q \in \mathscr{Q}} \Big(\big(\bigwedge_{A \in Q} Qy \bigvee_{i \in A} y{=}x_i \big) \wedge \big(\bigwedge_{A \notin Q} \neg Qy \bigvee_{i \in A} y{=}x_i \big) \Big) \to \bigvee_{A \in Q'} \Big(\bigwedge_{i \in A} Px_i \wedge \bigwedge_{i \notin A} \neg Px_i \Big) \Big]$$

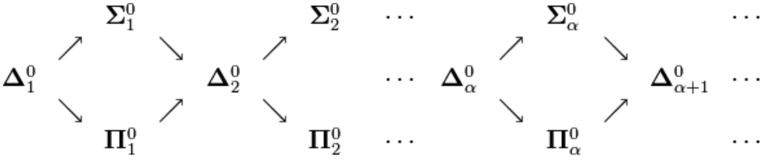
Borel sets

Definition

The sets generated by sets of the form $A_k = \{ X \subseteq \omega \mid k \in X \}$ by complements, finite unions and finite intersections are the **basic open sets.** *The sets generated from the basic open ones by arbitrary unions are the* **open sets.**

Definition

The sets generated by the open sets by countable unions, complements and countable intersections are the **Borel sets***.*



Assume now all quantifiers are local quantifiers on ω.

Theorem (Lopez-Escobar)

A quantifier is **Borel** *and* **permutation invariant** *iff it is definable in* $\mathscr{L}_{\omega_1\omega}$*.*

Indicates a strong connection between $\mathscr{L}_{\omega_1\omega}$ and Borel quantifiers.

FALSE

Q is Borel and $\mathrm{Aut}(\mathscr{Q})$ *invariant iff Q is definable in* $\mathscr{L}_{\omega_1\omega}(\mathscr{Q})$*.*

Let $A \subseteq \omega$ be infinite and coinfinite and $Q' = \{ A \}$ then Q^A is $\mathrm{Aut}(Q')$ invariant, but not definable in $\mathscr{L}_{\omega_1\omega}(Q')$.

The full symmetric group S_∞ on ω is naturally equipped with a topology: $G_{\bar{a},\bar{b}} = \{ g \mid g\bar{a} = \bar{b} \}$ as basic open sets.

Theorem (Vaught and E/Schlicht)

If F is the the family of orbits of a closed subgroup G then every G-invariant quantifier is $\mathscr{L}_{\omega_1\omega}(F)$-definable.

Theorem (E/Schlicht)

Let G be closed, then there is a closed **binary** *quantifier Q such that $G = \mathrm{Aut}(Q)$. Thus $\mathrm{Aut}(\mathrm{BInv}(G)) = G$ for closed groups G.*

Theorem (E/Schlicht)

Let Q_0 be a clopen quantifier. Then Q is Borel and $\mathrm{Aut}(Q_0)$ invariant iff it is definable in $\mathscr{L}_{\omega_1\omega}(Q_0)$.

Thanks