Satisfaction classes in nonstandard models of first-order arithmetic

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Arithmetic

Arithmetic is the part of mathematics considering natural numbers \((0,1,2,3,\ldots)\) and the two operations addition \((+)\) and multiplication \((\cdot)\).

For example, given a natural number \(n\), finding natural numbers \(d, q > 1\) such that

\[
 n = d \cdot q
\]

is a problem in arithmetic.
A property is ‘first-order’ if it can be expressed by only quantifying over individuals and not sets.

The statement

‘for all $n > 1$ there are natural numbers $p$ and $m$ such that $p$ is prime and $n = p \cdot m$’

is a first-order statement.

The statement

‘in any set of natural numbers there is a least member’

is not a first-order statement.
First-order arithmetic

First-order arithmetic (also known as Peano Arithmetic, or PA) is a first-order axiomatisation of arithmetic.

- Non-logical language $\mathcal{L}_A : S, +, \cdot, 0$.
- Axioms:
  - Defining axioms for $S$, $+$, $\cdot$ and $0$,
  - Induction axioms saying that every *definable set* of natural numbers has a least member (infinitely many).

By the Gödel-Rosser incompleteness theorem there is no axiomatisation of true first-order arithmetic, but PA captures most of true first-order arithmetic.
A model is a set together with some extra structure in the form of constants, functions and predicates.

A theory is a set of first-order sentences (statements).

A model is a model of a theory if all statements in the theory is true about the model.

A model $M$ of first-order arithmetic is therefore a set with one constant $0^M$ and three functions $S^M$, $+^M$ and $\cdot^M$ where the axioms of PA are true.

If $\varphi$ is true about (in) $M$ we write $M \models \varphi$. 
The set of natural numbers $\mathbb{N}$ together with the ‘real’ successor, the ‘real’ addition, and the ‘real’ multiplication functions is a model of $\mathbb{PA}$. It is called the standard model of $\mathbb{PA}$.

True arithmetic is the theory of all sentences true in the standard model of $\mathbb{PA}$.

Any other model of $\mathbb{PA}$ is called ‘nonstandard.’
Truth inside the model

If $\mathcal{M}$ is a model of $\mathsf{PA}$, we know, by Tarski’s undefinability of truth, that there is no formula $\text{Tr}(x)$ in $\mathcal{L}_A$ such that

$$\mathcal{M} \models \text{Tr}(\overline{\varphi}) \iff \varphi$$

for all $\mathcal{L}_A$-sentences $\varphi$. Here $\overline{\cdot}$ is a (recursive) map from the formulas in $\mathcal{L}_A$ into the natural numbers.

What we can (clearly) do is to add an extra predicate to the language, $\Sigma$, such that

$$\mathcal{M} \models \Sigma(\overline{\varphi}) \iff \varphi$$

for all $\mathcal{L}_A$-sentences $\varphi$. 

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Nonstandard sentences

In a nonstandard model $\mathcal{M}$ of $\text{PA}$ there are nonstandard elements $a > \mathbb{N}$.

In any such model there exists nonstandard elements which the model “thinks” are formulas.

The formula

$$0 \neq 0 \lor (0 \neq 0 \lor (0 \neq 0 \lor \ldots \lor 0 \neq 0))$$

is a nonstandard formula if the dots represent a nonstandard number of repetition.

When is such a formula true?
Satisfaction classes

What do we want $\Sigma$ to say about nonstandard sentences $\varphi \in \mathcal{M} \setminus \mathbb{N}$?

We want, at least, the following to be true for all standard and nonstandard sentences $\varphi$:

$$
\Sigma(\neg \varphi) \iff \neg \Sigma(\varphi)
$$

$$
\Sigma(\varphi \lor \psi) \iff (\Sigma(\varphi) \lor \Sigma(\psi))
$$

$$
\Sigma(\exists x \varphi(x)) \iff \exists x \Sigma(\varphi(x)).
$$

Such a $\Sigma$ is called a satisfaction class.
Recursive saturation

In fact, not all models of \( \mathbb{PA} \) have satisfaction classes. It turns out that for the countable models it is exactly the recursively saturated models which do admit satisfaction classes.

A model is recursively saturated if it is ‘big’ and ‘homogeneous’ (in some technical sense = all recursive types are realized)

Every countable recursively saturated model of \( \mathbb{PA} \) has \( 2^{\aleph_0} \) different satisfaction classes.
Define $\models_M \varphi$ for a nonstandard (or standard) sentence $\varphi$ to mean that every satisfaction class includes $\varphi$.

For standard sentences $\varphi$ we have

$$M \models \varphi \iff \models_M \varphi.$$  

It turns out that $\models_M \varphi$ iff for some finite approximation $\psi$ of $\varphi$, $M \models \psi$.

A finite approximation of $\varphi$ is a standard formula which you get by replacing some subformulas of $\varphi$ by predicate symbols.
Example

Let $\varphi_a$ be the formula

$$0 \neq 0 \lor (0 \neq 0 \lor (0 \neq 0 \lor \ldots \lor 0 \neq 0))$$

with a subformulas $0 \neq 0$.

The finite approximations of this formula are (more or less):

$$P_{\varphi_a}$$

$$0 \neq 0 \lor P_{\varphi_{a-1}}$$

$$0 \neq 0 \lor (0 \neq 0 \lor P_{\varphi_{a-2}})$$

$$\vdots$$
Example (cont)

Since we can interpret the predicate symbols $P_{\varphi_{a-k}}$ either as ‘true’ or ‘false’ we can make all these finite approximations true or all of them true. Therefore

$$\models_M \varphi_a \quad \text{and} \quad \models_M \neg \varphi_a$$

for any nonstandard $a \in M$.

Also

$$\models_M S(S(\ldots S(0) \ldots)) = a \quad \text{and} \quad \models_M S(S(\ldots S(0) \ldots)) \neq a$$

for any nonstandard $a \in M$ and any nonstandard number of function symbols $S$.

These sentences, which we can intuitively decide, but which are not decided by $\models_M$ are called ‘pathologies.’
# Removal of pathologies

<table>
<thead>
<tr>
<th>Pathology</th>
<th>Solution</th>
<th>Con</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S(\ldots S(0) \ldots) = a )</td>
<td>Axioms: ( Tr_{At} )</td>
<td>Yes</td>
</tr>
<tr>
<td>( 0 \neq 0 \lor \ldots \lor 0 \neq 0 )</td>
<td>Axioms: ( Tr_{\Delta_0} )</td>
<td>Yes</td>
</tr>
<tr>
<td>( (\varphi \land \neg \varphi) \lor \ldots \lor (\varphi \land \neg \varphi) )</td>
<td>Rule: Prop</td>
<td>?</td>
</tr>
<tr>
<td>( \exists v_0 v_1 \ldots v_a 0 \neq 0 )</td>
<td>Axioms: ( Tr_{\Sigma_1} )</td>
<td>Yes</td>
</tr>
<tr>
<td>( (\exists v_0 v_1 \ldots v_a \varphi) \leftrightarrow \neg \varphi )</td>
<td>Rules: ( \exists I_\infty, M_\infty )-rule</td>
<td>?</td>
</tr>
<tr>
<td>( (\exists v_0 \forall v_1 \ldots \exists v_{2a} \varphi) \leftrightarrow \neg \varphi )</td>
<td>Rule: Skolem</td>
<td>?</td>
</tr>
<tr>
<td></td>
<td>Rule: Pred</td>
<td>?*</td>
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* It is known that if we also include all nonstandard instances of axioms of \( \mathsf{PA} \) then the consistency of \( \mathsf{Pred} \) depends on the theory of \( M \).
We know that the consistency of any of these logics can only depend on the first-order theory of $\mathcal{M}$, i.e., given a completion $T$ of $\mathit{PA}$, either the logic is consistent in all countable recursively saturated models or in none.

This is due to the resplendency of all countable recursively saturated models.
Given a satisfaction class $\Sigma$ we can define the valuation $\text{val}_\Sigma$ to be a map from the nonstandard closed terms into $\mathcal{M}$ by saying $\text{val}_\Sigma(t) = a$ iff $\Sigma(t = a)$.

Why do these valuations take values in $\mathcal{M}$?

One answer: A rather “ugly looking” axiom in the proof theory of satisfaction classes.
A free satisfaction class $\Sigma$ is one where this axiom is removed and therefore $\text{val}_\Sigma$ can take values in some $\mathcal{N} \supseteq \mathcal{M}$.

Given a free satisfaction class we can define the model $\mathcal{M}_\Sigma$ which is the model of values of all closed, standard and nonstandard, terms. We know that $\mathcal{M} \subseteq \mathcal{M}_\Sigma$.

Natural question: Which models do arise in this way?

A free satisfaction class is free of existential assumptions for nonstandard terms, since we could have $\neg \exists x(t = x)$ for some nonstandard term $t$. 