Satisfaction classes in nonstandard models of first-order arithmetic

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Arithmetic

Arithmetic is the part of mathematics considering natural numbers (0,1,2,3,…) and the two operations addition ($+$) and multiplication ($\cdot$).

For example, given a natural number n, finding natural numbers $d, q > 1$ such that

$$n = d \cdot q$$

is a problem in arithmetic.

First-order

A property is ‘first-order’ if it can be expressed by only quantifying over individuals and not sets.

The statement

> ‘for all $n > 1$ there are natural numbers p and m such that p is prime and $n = p \cdot m$’

is a first-order statement.

The statement

> ‘in any set of natural numbers there is a least member’

is *not* a first-order statement.

First-order arithmetic

First-order arithmetic (also known as Peano Arithmetic, or $\mathtt{PA}$) is a first-order axiomatisation of arithmetic.

- Non-logical language $\mathcal{L}_A : \mathsf{S}, +, \cdot, 0$.
- Axioms:
  - Defining axioms for S, $+$, $\cdot$ and 0,
  - Induction axioms saying that every *definable set* of natural numbers has a least member (infinitely many).

By the Gödel-Rosser incompleteness theorem there is no axiomatisation of true first-order arithmetic, but $\mathtt{PA}$ captures most of true first-order arithmetic.

Model

A model is a set together with some extra structure in the form of constants, functions and predicates.

A theory is a set of first-order sentences (statements).

A model is a model of a theory if all statements in the theory is true about the model.

A model $\mathfrak{M}$ of first-order arithmetic is therefore a set with one constant $0^{\mathfrak{M}}$ and three functions $\mathsf{S}^{\mathfrak{M}}$, $+^{\mathfrak{M}}$ and $\cdot^{\mathfrak{M}}$ where the axioms of $\mathtt{PA}$ are true.

If φ is true about (in) $\mathfrak{M}$ we write $\mathfrak{M} \models \varphi$.

Nonstandard

The set of natural numbers $\mathbb{N}$ together with the ‘real’ successor, the ‘real’ addition, and the ‘real’ multiplication functions is a model of $\mathtt{PA}$. It is called the standard model of $\mathtt{PA}$.

True arithmetic is the theory of all sentences true in the standard model of $\mathtt{PA}$.

Any other model of $\mathtt{PA}$ is called ‘nonstandard.’

Truth inside the model

If $\mathfrak{M}$ is a model of $\mathtt{PA}$, we know, by Tarski's undefinability of truth, that there is no formula $\mathtt{Tr}(x)$ in $\mathcal{L}_A$ such that

$$\mathfrak{M} \models \mathtt{Tr}(\ulcorner\varphi\urcorner) \leftrightarrow \varphi$$

for all $\mathcal{L}_A$-sentences φ. Here $\ulcorner\cdot\urcorner$ is a (recursive) map from the formulas in $\mathcal{L}_A$ into the natural numbers.

What we *can* (clearly) do is to add an extra predicate to the language, Σ, such that

$$\mathfrak{M} \models \Sigma(\ulcorner\varphi\urcorner) \leftrightarrow \varphi$$

for all $\mathcal{L}_A$-sentences φ.

Nonstandard sentences

In a nonstandard model $\mathfrak{M}$ of $\mathtt{PA}$ there are nonstandard elements $a > \mathbb{N}$.

In any such model there exists nonstandard elements which the model "thinks" are formulas.

The formula

$$0 \neq 0 \vee (0 \neq 0 \vee (0 \neq 0 \vee \ldots \vee 0 \neq 0))$$

is a nonstandard formula if the dots represent a nonstandard number of repetition.

When is such a formula true?

Satisfaction classes

What do we want Σ to say about nonstandard sentences $\varphi \in \mathfrak{M} \setminus \mathbb{N}$?

We want, at least, the following to be true for *all* standard and nonstandard sentences φ:

$$\begin{aligned} \Sigma(\neg\varphi) &\iff \neg\Sigma(\varphi) \\ \Sigma(\varphi \vee \psi) &\iff \big(\Sigma(\varphi) \vee \Sigma(\psi)\big) \\ \Sigma(\exists x \varphi(x)) &\iff \exists x \Sigma(\varphi(x)). \end{aligned}$$

Such a Σ is called a **satisfaction class.**

Recursive saturation

In fact, not all models of $\mathtt{PA}$ have satisfaction classes. It turns out that for the countable models it is exactly the recursively saturated models which do admit satisfaction classes.

A model is recursively saturated if it is 'big' and 'homogeneous' (in some technical sense $=$ all recursive types are realized)

Every countable recursively saturated model of $\mathtt{PA}$ has $2^{\aleph_0}$ different satisfaction classes.

Consistency

Define $\models_{\mathfrak{M}} \varphi$ for a nonstandard (or standard) sentence φ to mean that every satisfaction class includes φ.

For standard sentences φ we have

$$\mathfrak{M} \models \varphi \quad \text{iff} \quad \models_{\mathfrak{M}} \varphi.$$

It turns out that $\models_{\mathfrak{M}} \varphi$ iff for some finite approximation ψ of φ, $\mathfrak{M} \models \psi$.

A finite approximation of φ is a standard formula which you get by replacing some subformulas of φ by predicate symbols.

Example

Let φ_a be the formula

$$0 \neq 0 \vee (0 \neq 0 \vee (0 \neq 0 \vee \ldots \vee 0 \neq 0))$$

with a subformulas $0 \neq 0$.

The finite approximations of this formula are (more or less):

$$P_{\varphi_a}$$

$$0 \neq 0 \vee P_{\varphi_{a-1}}$$

$$0 \neq 0 \vee (0 \neq 0 \vee P_{\varphi_{a-2}})$$

$$\vdots$$

Example (cont)

Since we can interpret the predicate symbols $P_{\varphi_{a-k}}$ either as 'true' or 'false' we can make all these finite approximations true or all of them true.
Therefore

$$\not\models_{\mathfrak{M}} \varphi_a \quad \text{and} \quad \not\models_{\mathfrak{M}} \neg\varphi_a$$

for any nonstandard $a \in \mathfrak{M}$.
Also

$$\not\models_{\mathfrak{M}} \mathsf{S}(\mathsf{S}(\ldots \mathsf{S}(0)\ldots)) = a \quad \text{and} \quad \not\models_{\mathfrak{M}} \mathsf{S}(\mathsf{S}(\ldots \mathsf{S}(0)\ldots)) \neq a$$

for any nonstandard $a \in \mathfrak{M}$ and any nonstandard number of function symbols S.
These sentences, which we can intuitively decide, but which are not decided by $\models_{\mathfrak{M}}$ are called 'pathologies.'

Removal of pathologies

Pathology	Solution	Con
$\mathsf{S}(\ldots\mathsf{S}(0)\ldots) = a$	Axioms: $\mathtt{Tr}_{\mathsf{At}}$	Yes
$0 \neq 0 \vee \ldots \vee 0 \neq 0$	Axioms: $\mathtt{Tr}_{\Delta_0}$	Yes
$(\varphi \wedge \neg\varphi) \vee \ldots \vee (\varphi \wedge \neg\varphi)$	Rule: Prop	**?**
$\exists \mathsf{v}_0 \mathsf{v}_1 \ldots \mathsf{v}_a\, 0 \neq 0$	Axioms: $\mathtt{Tr}_{\Sigma_1}$	Yes
$(\exists \mathsf{v}_0 \mathsf{v}_1 \ldots \mathsf{v}_a\, \varphi) \leftrightarrow \neg\varphi$	Rules: $\exists I_\infty$, $\mathfrak{M}_\infty$-rule	?
	Rule: Pred	?*
$(\exists \mathsf{v}_0 \forall \mathsf{v}_1 \ldots \exists \mathsf{v}_{2a}\, \varphi) \leftrightarrow \neg\varphi$	Rule: Skolem	?
	Rule: Pred	?*

* It is known that if we also include all nonstandard instances of axioms of $\mathtt{PA}$ then the consistency of Pred depends on the theory of $\mathfrak{M}$.

Resplendency

We know that the consistency of any of these logics can *only* depend on the first-order theory of $\mathfrak{M}$, i.e., given a completion T of $\mathtt{PA}$, either the logic is consistent in all countable recursively saturated models or in none.

This is due to the resplendency of all countable recursively saturated models.

Atomics

Given a satisfaction class Σ we can define the valuation $\mathtt{val}_\Sigma$ to be a map from the nonstandard closed terms into $\mathfrak{M}$ by saying $\mathtt{val}_\Sigma(t) = a$ iff $\Sigma(t = a)$.

Why do these valuations take values in $\mathfrak{M}$?

One answer: A rather “ugly looking” axiom in the proof theory of satisfaction classes.

Free satisfaction classes

A free satisfaction class Σ is one where this axiom is removed and therefore $\texttt{val}_\Sigma$ can take values in some $\mathfrak{N} \supseteq \mathfrak{M}$.

Given a free satisfaction class we can define the model $\mathfrak{N}_\Sigma$ which is the model of values of all closed, standard and nonstandard, terms. We know that $\mathfrak{M} \subseteq \mathfrak{N}_\Sigma$.

Natural question: Which models do arise in this way?

A free satisfaction class is free of existential assumptions for nonstandard terms, since we could have $\neg\exists x(t = x)$ for some nonstandard term t.