Notions of resplendency for logics stronger than first-order logic

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These slides are available at:

`http://engstrom.morot.org`

Preliminaries

- All languages will be recursive, all extensions of languages recursive extensions.
- We will restrict ourselfs to models of arithmetic, even though many of the results hold for arbitrary models.

Plan

We introduce three new variations on resplendency and recursive saturation:

- *transcendence*, resplendency for a specifi c infi nitary language;
- *subtranscendence*, subresplendency for the same language; and
- *recursive standard saturation*, recursive saturation for a language with a standard predicate.

Recursive saturation...

- A *type* $p(x, a)$ over a model M is a set of formulas, with parameter $a \in M$, consistent with the theory of (M, a).
- M is *recursively saturated* if all recursive types over M are realized.
- Any model M has an elementary extension of the same cardinality which is recursively saturated.

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- $\mathrm{SSy}(M)$ is the standard system of $M \models \mathrm{PA}$, i.e., the collection of standard parts of parameter defi nable sets; i.e., the collection of all $\{\, n \in \omega \mid M \models \varphi(n,a) \,\}$, where $a \in M$.
- If M is recursively saturated then any type $p(x,a) \in \mathrm{SSy}(M)$ over M is realized in M.

Resplendency

- A model M is *resplendent* if for every $a \in M$, every $\mathscr{L}^+ \supseteq \mathscr{L}(a)$, and every recursive T in $\mathscr{L}^+$ such that $\mathrm{Th}(M, a) + T$ is consistent there is an expansion M^+ of M satisfying T.
- Every countable recursively saturated model is resplendent, and every resplendent model is recursively saturated.

Stronger logics

- If $p(x, a)$ is a type, $p\uparrow$ stands for the infinitary sentence expressing that $p(x, a)$ is omitted.
- If T_0 is a first-order theory, T also first-order in an extended language, and $p(\bar{x})$ a type in the same language as T; then $\mathrm{SatCon}(T + p\uparrow / T_0)$ holds iff there is an ω-saturated model of T_0 with an expansion satisfying $T + p\uparrow$.

A strong saturation property...

- If a recursively saturated model, M, satisfi es that for all $T_0, T, p(x) \in \mathrm{SSy}(M)$ such that $\mathrm{SatCon}(T + p{\uparrow}/T_0)$ there is a completion $T_c \in \mathrm{SSy}(M)$ of T such that $\mathrm{SatCon}(T_c + p{\uparrow}/T_0)$; then M is said to be SatCon-saturated.
- Every model has a SatCon-saturated elementary extension of the same cardinality.

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- Every SatCon-saturated countable model M is *transcendent*:

  For all $a \in M$ and all $T, p(x) \in \mathrm{SSy}(M)$, in an extension of the language of (M, a), if $\mathrm{SatCon}(T + p{\uparrow}/\mathrm{Th}(M, a))$ then there is an expansion M^+ of M such that $M^+ \models T + p{\uparrow}$.

- Is a transcendent model SatCon-saturated?

Subresplendency

- A model M is *subresplendent* if for every $a \in M$, every recursive extension $\mathscr{L}^+$ of $\mathscr{L}(a)$ and every recursive T in $\mathscr{L}^+$ such that $\mathrm{Th}(M, a) + T$ is consistent there is an elementary submodel $a \in N \prec M$ and an expansion N^+ of N satisfying T.
- A model is subresplendent iff it is recursively saturated.

β-saturation

- A recursively saturated model is β*-saturated* if $\mathrm{SSy}(M)$ is a β-model.
- All β-saturated models are subtranscendent:

  For all $a \in M$ and all $T, p(x) \in \mathrm{SSy}(M)$, if $\mathrm{Th}(M, a) + T + p\!\uparrow$ is consistent then there is an elementary submodel $a \in N \prec M$ and an expansion N^+ of N satisfying $T + p\!\uparrow$.
- In fact; a model is subtranscendent iff it is β-saturated.

The standard predicate

- The standard predicate, st, is the predicate of standard numbers.
- No model (M, st) is recursively saturated since the type

$$\{\, x > n \wedge \mathrm{st}(x) \mid n \in \omega \,\}$$

is omitted.

Standard recursive saturation

- A *standard type over* M is a type over (M, st) such that there is an ω-saturated elementary extension of M realizing the type.
- A model is *recursively standard saturated* (rec std sat) if all recursive standard types are realized.
- Any type over M (in which st is not mentioned) is a standard type.

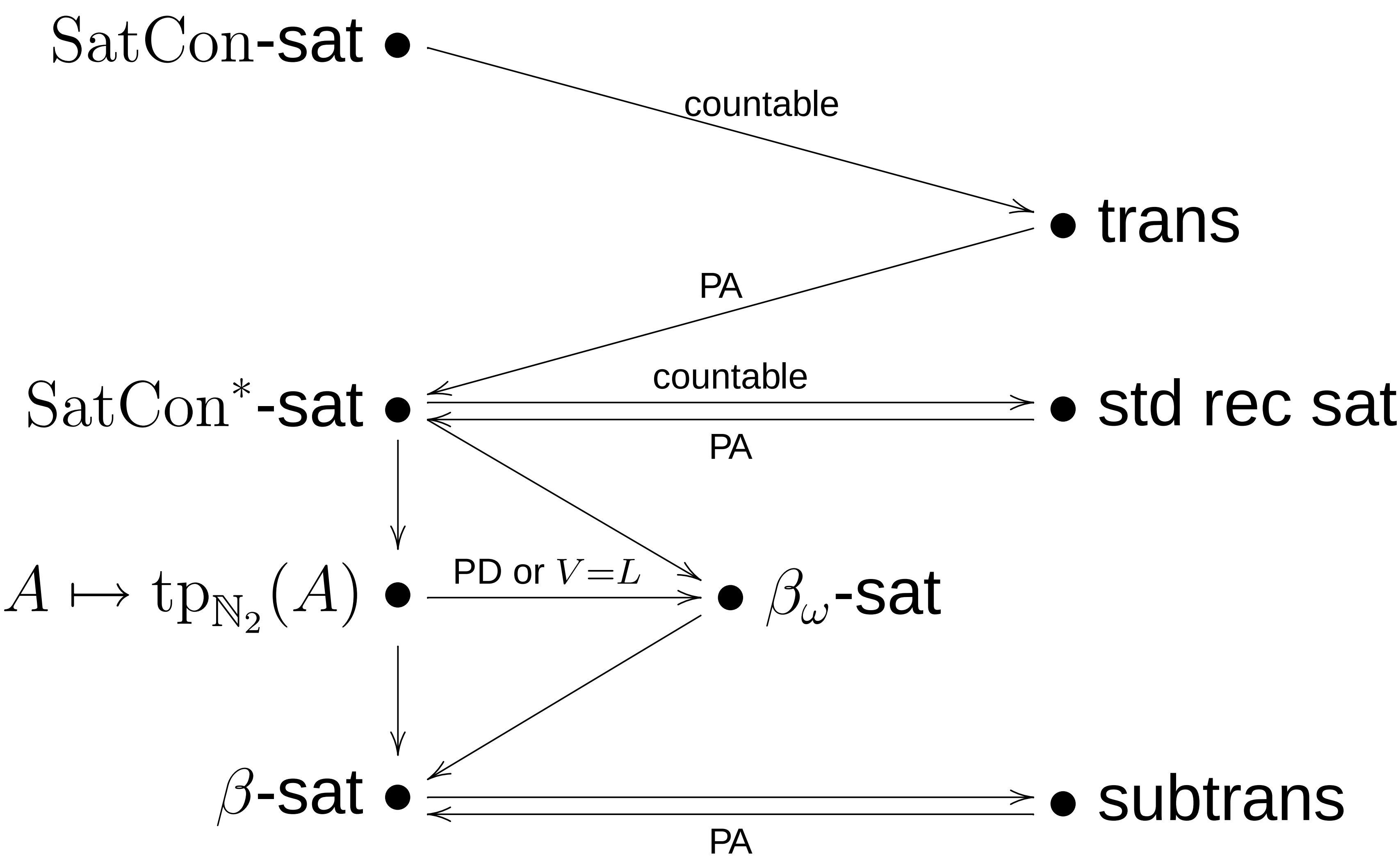
An equivalence

A countable recursively saturated model is rec std sat iff

> for all standard types $p(x, a) \in \mathrm{SSy}(M)$ over M there is a complete standard type $q(x, a) \in \mathrm{SSy}(M)$ extending $p(x, a)$.

We call this property SatCon^*-saturated.

Summary


SatCon-sat
countable
trans
PA
SatCon*-sat
countable
std rec sat
PA
$A \mapsto \mathrm{tp}_{\mathbb{N}_2}(A)$
PD or $V=L$
β_ω-sat
β-sat
subtrans
PA

The end

That’s all folks!