Notions of resplendency for logics stronger than first-order logic

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These slides are available at:
http://engstrom.morot.org
Preliminaries

- All languages will be recursive, all extensions of languages recursive extensions.
- We will restrict ourselves to models of arithmetic, even though many of the results hold for arbitrary models.
Plan

We introduce three new variations on resplendency and recursive saturation:

- *transcendence*, resplendency for a specific infinitary language;
- *subtranscendence*, subresplendency for the same language; and
- *recursive standard saturation*, recursive saturation for a language with a standard predicate.
A type $p(x, a)$ over a model $M$ is a set of formulas, with parameter $a \in M$, consistent with the theory of $(M, a)$.

$M$ is recursively saturated if all recursive types over $M$ are realized.

Any model $M$ has an elementary extension of the same cardinality which is recursively saturated.
SSy(M) is the standard system of $M \models \text{PA}$, i.e., the collection of standard parts of parameter definable sets; i.e., the collection of all $\{ n \in \omega \mid M \models \varphi(n, a) \}$, where $a \in M$.

If $M$ is recursively saturated then any type $p(x, a) \in \text{SSy}(M)$ over $M$ is realized in $M$. 

...continued
A model $M$ is *resplendent* if for every $a \in M$, every $\mathcal{L}^+ \supseteq \mathcal{L}(a)$, and every recursive $T$ in $\mathcal{L}^+$ such that $\text{Th}(M, a) + T$ is consistent there is an expansion $M^+$ of $M$ satisfying $T$.

Every countable recursively saturated model is resplendent, and every resplendent model is recursively saturated.
Stronger logics

- If $p(x, a)$ is a type, $p^\uparrow$ stands for the infinitary sentence expressing that $p(x, a)$ is omitted.

- If $T_0$ is a first-order theory, $T$ also first-order in an extended language, and $p(\bar{x})$ a type in the same language as $T$; then $\text{SatCon}(T + p^\uparrow/T_0)$ holds iff there is an $\omega$-saturated model of $T_0$ with an expansion satisfying $T + p^\uparrow$. 
A strong saturation property...

- If a recursively saturated model, $M$, satisfies that for all $T_0, T, p(x) \in SSy(M)$ such that $SatCon(T + p/T_0)$ there is a completion $T_c \in SSy(M)$ of $T$ such that $SatCon(T_c + p/T_0)$; then $M$ is said to be $SatCon$-saturated.

- Every model has a $SatCon$-saturated elementary extension of the same cardinality.
Every SatCon-saturated countable model $M$ is *transcendent*:

For all $a \in M$ and all $T, p(x) \in SSy(M)$, in an extension of the language of $(M, a)$, if SatCon$(T + p^\uparrow/\text{Th}(M, a))$ then there is an expansion $M^+$ of $M$ such that $M^+ \models T + p^\uparrow$.

Is a transcendent model SatCon-saturated?
Subresplendency

- A model $M$ is subresplendent if for every $a \in M$, every recursive extension $\mathcal{L}^+$ of $\mathcal{L}(a)$ and every recursive $T$ in $\mathcal{L}^+$ such that $\text{Th}(M, a) + T$ is consistent there is an elementary submodel $a \in N \prec M$ and an expansion $N^+$ of $N$ satisfying $T$.

- A model is subresplendent iff it is recursively saturated.
A recursively saturated model is \( \beta \)-saturated if \( \text{SSy}(M) \) is a \( \beta \)-model.

All \( \beta \)-saturated models are subtranscendent:

For all \( a \in M \) and all \( T, p(x) \in \text{SSy}(M) \), if \( \text{Th}(M, a) + T + p \uparrow \) is consistent then there is an elementary submodel \( a \in N \prec M \) and an expansion \( N^+ \) of \( N \) satisfying \( T + p \uparrow \).

In fact; a model is subtranscendent iff it is \( \beta \)-saturated.
The standard predicate

The standard predicate, $st$, is the predicate of standard numbers.

No model $(M, st)$ is recursively saturated since the type

$$\{ x > n \land st(x) \mid n \in \omega \}$$

is omitted.
Standard recursive saturation

- A *standard type over* $\mathcal{M}$ is a type over $(\mathcal{M}, st)$ such that there is an $\omega$-saturated elementary extension of $\mathcal{M}$ realizing the type.

- A model is *recursively standard saturated* (rec std sat) if all recursive standard standard types are realized.

- Any type over $\mathcal{M}$ (in which $st$ is not mentioned) is a standard type.
An equivalence

A countable recursively saturated model is recursively saturated if and only if for all standard types $p(x, a) \in S\text{Sy}(M)$ over $M$ there is a complete standard type $q(x, a) \in S\text{Sy}(M)$ extending $p(x, a)$.

We call this property $\text{SatCon}^*$-saturated.
Summary

SatCon-sat \bullet
countable
\rightarrow
trans

SatCon*-sat \bullet
countable
PA
\rightarrow
std rec sat

A \leftrightarrow tp_{N_2}(A)
PD or V=L
\rightarrow
\beta_\omega\text{-sat}

\beta\text{-sat} \bullet
PA
\rightarrow
subtrans

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The end

That’s all folks!