Dependence Logic with Generalized Quantifiers: Axiomatizations

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Dependence logic

$$\forall x \exists y \forall z \exists w \, Rxyzw$$

$$\downarrow$$

$$\forall z$$

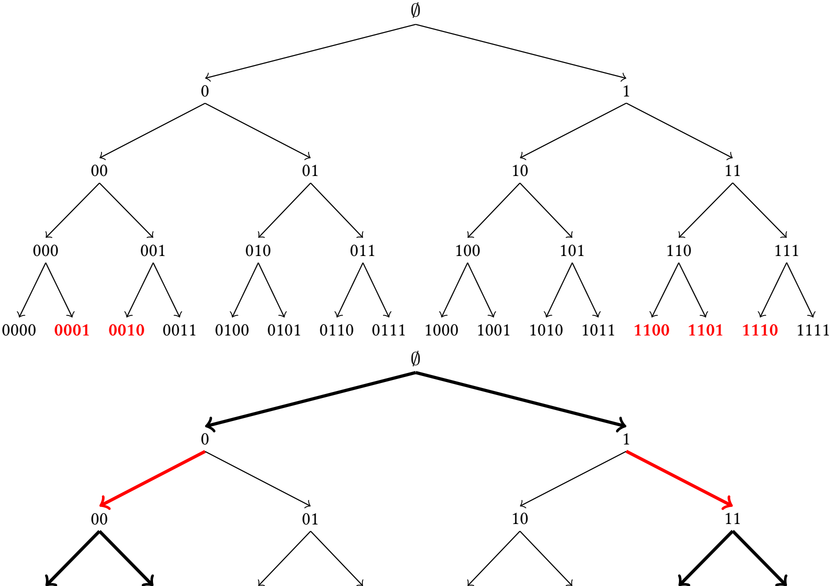
$$\downarrow$$

$$\exists w$$

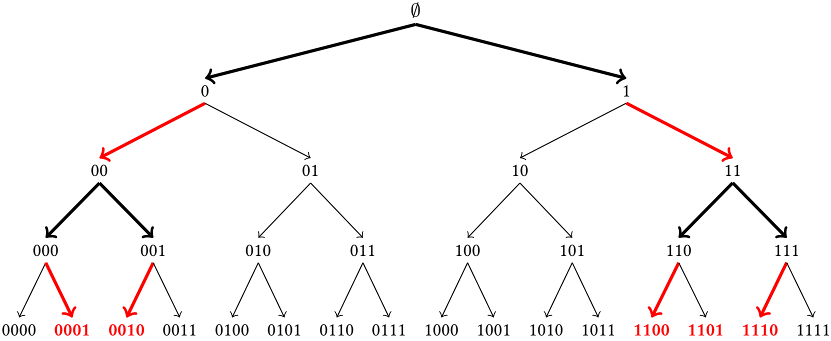
$$\begin{pmatrix} \forall x \, \exists y \\ \forall z \, \exists w \end{pmatrix} Rxyzw$$

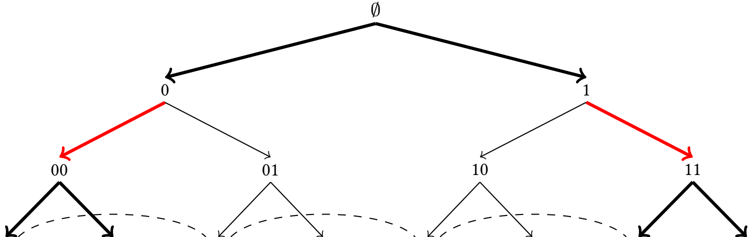
$$\begin{array}{cc} \forall z & \forall x \\ \downarrow & \downarrow \\ \exists w & \exists y \end{array}$$

Domain $\{ 0, 1 \}$. $\forall x \exists y \forall z \exists w Rxyzw$



Domain $\{\,0,1\,\}$. $\begin{pmatrix}\forall x\,\exists y\\ \forall z\,\exists w\end{pmatrix}$ $Rxyzw$





x	y	z	w
0	0	0	1
0	0	1	0
1	1	0	0
1	1	1	0

$\not\models \; =\!(z, w)$

x	y	z	w
0	0	0	1
0	0	1	0
1	1	0	1
1	1	1	0

$\models \; =\!(z, w)$

$$\begin{pmatrix} \forall x \exists y \\ \forall z \exists w \end{pmatrix} Rxyzw \equiv \forall x \exists y \forall z \exists w (=\!(z, w) \wedge Rxyzw)$$

Definition

X a **team** = set of assignments.

Generalized quantifiers

- **Generalized quantifier**: Q a class of structures (one unary relation).
- $M, s \vDash Qx\,\phi$ iff $(M, \phi^{M,s}) \in Q$.
- $Q_M = \{ R \mid (M, R) \in Q \}$.

- $\mathrm{FO}(Q)$ is FOL extended with expressions $Qx\,\phi$.
- $\exists$ and $\forall$ can be interpreted as generalized quantifiers.
- We will **only consider (non-trivial) monotone increasing quantifiers**: If $A \subseteq B$ and $A \in Q_M$ then $B \in Q_M$.
- $\check{Q} = \{ (M, R^c) \mid (M, R) \notin Q \}$. $\neg Qx\neg\phi \equiv \check{Q}x\,\phi$.

Dependence Logic with Q

- $D(Q)$ is $\phi ::= \gamma \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x \phi \mid \forall x \phi \mid Qx \phi$, γ an $\mathrm{FO}(Q)$ formula or dependence atom.
- $M \vDash \sigma$ iff $M, \{\emptyset\} \vDash \sigma$.

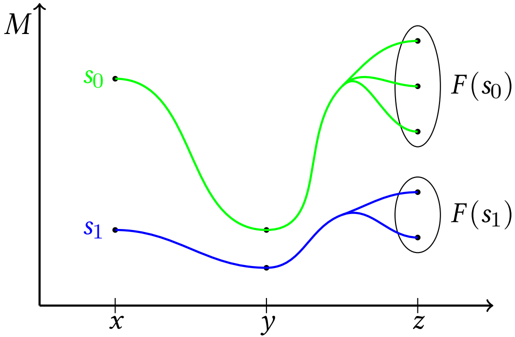
- $M, X \vDash \gamma$ if for all $s \in X$: $M, s \vDash \gamma$, γ an $\mathrm{FO}(Q)$ formula.
- $M, X \vDash \phi \wedge \psi$ iff $M, X \vDash \phi$ and $M, X \vDash \psi$.
- $M, X \vDash \phi \vee \psi$ iff there are $Y \cup Z = X$ such that $M, Y \vDash \phi$ and $M, Z \vDash \psi$.

Quantifiers in dependence logic

- $M, X \vDash Qx\,\phi$ iff there is $F : X \to Q_M$ such that $M, X[F/x] \vDash \phi$.

$$X[F/x] = \{\, s[a/x] \mid s \in X, a \in F(s) \,\}.$$

Example: $M, \{s_0, s_1\} \vDash Qz\,Rxyz$



- Note: using set-valued Fs corresponds to **non-deterministic** strategies.

Properties of Dependence logic

- $M, \emptyset \vDash \phi$
- **Downwards closure:** If $Y \subseteq X$ and $M, X \vDash \phi$ then $M, Y \vDash \phi$.
- $D(Q) \equiv \mathrm{ESO}(Q)$ (E / Kontinen)
- Branching of generalized quantifiers (p.o. quantifier prefixes) may be expressed in $D(Q)$.

Theorem: Normal form for $D(Q, \check{Q})$
Every $D(Q, \check{Q})$ formula is equivalent to one of the form:

$$\mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m \exists y_1 \ldots \exists y_n \Big(\bigwedge_{1 \leq i \leq n} =(\overline{x}^i, y_i) \wedge \theta \Big),$$

where $\mathcal{H}^i$ is either Q, $\check{Q}$ or $\forall$, and θ is a quantifier-free FO formula.

Axiomatizations

Dependence logic

- Dependence relations can be axiomatized (Armstong).
- Dependence logic has the same strength as ESO.
- The relation $\Gamma \vDash \phi$ is not r.e.
- Restricting to ϕ's without dependence atoms gives an r.e. entailment relation.
- An explicit axiomatization has been given by Kontinen and Väänänen.

Idea:

- Construct a natural deduction system in which the normal form can be derived.
- Allow dependencies in normal forms to be replaced by **finite approximations**.
- Show that in enough models (recursively saturated) the set of finite approximations is equivalent to the original sentence.

Axiomatizing $D(Q, \check{Q})$ I: General rules

First: Some rules sound for **any interpretation** of Q (monotone increasing).

- Standard rules for $\mathrm{FO}(Q, \check{Q})$ formulas.
- Standard rules for conjunction, existential quantifier, and universal quantifier.
- Commutativity, associativity and monotonicity of disjunction.
- Monotonicity, extending scope, and renaming of bound variables for Q and $\check{Q}$.
- Duality of $\check{Q}$ with respect to $\mathrm{FO}(Q, \check{Q})$ formulas.

Axiomatizing $D(Q, \check{Q})$ II: Dependence related rules

- Unnesting:

$$\frac{=(t_1, ..., t_n)}{\exists z(=(t_1, ..., z, ..., t_n) \wedge z = t_i)}$$

where z is a new variable.

- Dependence distribution:

$$\frac{\exists y_1 \ldots \exists y_n(\bigwedge_{1 \leq j \leq n} =(\bar{z}^j, y_j) \wedge \phi) \vee \exists y_{n+1} \ldots \exists y_m(\bigwedge_{n+1 \leq j \leq m} =(\bar{z}^j, y_j) \wedge \psi}{\exists y_1 \ldots \exists y_m(\bigwedge_{1 \leq j \leq m} =(\bar{z}^j, y_j) \wedge (\phi \vee \psi))}$$

where ϕ and ψ are quantifier free FO formulas.

- Dependence introduction:

$$\frac{\exists x \mathcal{H} y \phi}{\mathcal{H} y \exists x (=(\bar{z}, x) \wedge \phi)}$$

where $\bar{z}$ lists the variables in $\mathrm{FV}(\phi) - \{x, y\}$ and $\mathcal{H} \in \{\forall, Q, \check{Q}\}$.

Approximations

Suppose σ is in normal form:

$$\mathcal{H}^1 x_1 \dots \mathcal{H}^m x_m \exists y_1 \dots \exists y_n \Big(\bigwedge_{1 \le i \le n} {=}(\bar{x}^i, y_i) \wedge \theta(\bar{x}, \bar{y}) \Big).$$

Let $A^k \sigma$ be

$$\forall \bar{x}_1 \exists \bar{y}_1 \dots \forall \bar{x}_k \exists \bar{y}_k \Big(\bigwedge_{1 \le j \le k} R(\bar{x}_j) \to \bigwedge_{1 \le j \le k} \theta(\bar{x}_j, \bar{y}_j) \wedge \bigwedge_{\substack{1 \le i \le n \\ 1 \le j, j' \le k}} (\bar{x}^i_j = \bar{x}^i_{j'} \to y_{i,j} = y_{i,j'}) \Big)$$

Let $B\sigma$ be

$$\mathcal{H}^1 x_1 \dots \mathcal{H}^m x_m R(x_1, \dots, x_m).$$

Axiomatizing $D(Q, \check{Q})$ III: The approximation rule

$$\dfrac{\sigma \qquad \begin{matrix} [B\sigma] & & [A^k\sigma] \\ & \ddots \quad \iddots & \\ & \psi & \end{matrix}}{\psi} \text{(Approx)}$$

where σ is a sentence in normal form, and R does not appear in ψ nor in any uncancelled assumptions in the derivation of ψ, except for $B\sigma$ and $A^k\sigma$.

Completeness for weak semantics

Let $\Gamma \vDash_w \phi$ mean that $\Gamma \vDash \phi$ for any monotone increasing (non-trivial) interpretation of Q (and $\check{Q}$ is interpreted as the dual of the interpretation of Q).

Theorem

This system is sound and complete wrt $\Gamma \vDash_w \phi$ where ϕ is $\mathrm{FO}(Q, \check{Q})$.

Uncountably many

- FO(Q_1) is axiomatizable, where Q_1 is "there exist uncountably many ...". (Kiesler)
- Add Keisler's rules for Q_1.

Define the **Skolem translation** $S\sigma$ of σ in normal form to be:

$$\mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m \theta(f_i(\bar{x}^i)/y_i).$$

- Replace the approximation rule with the following rule

$$\dfrac{\sigma \qquad \begin{array}{c}[S\sigma] \\ \vdots \\ \psi\end{array}}{\psi}\ \text{(Skolem)}$$

Theorem

This system is sound and complete wrt $\Gamma \models \phi$ where ϕ is FO($Q_1, \check{Q}_1$).

Conclusion

Extending dependence logic with generalized quantifiers is a natural and **stable** extension.

- The satisfaction relation is naturally defined when moving to non-deterministic strategies.
- $D(Q)$ properly extends both $\mathrm{FO}(Q)$ and D.
- $D(Q)$ is equivalent to $\mathrm{ESO}(Q)$.
- $D(Q, \check{Q})$ has a prenex normal form theorem.
- Two completeness results:
  - First wrt to weak semantics.
  - Second wrt to Q_1.
- Second result not fully satisfactory. Is it possible to get completeness for Q_1 using approximations instead of Skolem functions?

Thank you for your attention.

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