Dependence Logic with Generalized Quantifiers: Axiomatizations

WOLLIC 2013, Darmstadt

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August 21, 2013
Dependence logic
\[ \forall x \exists y \forall z \exists w Rxyzw \]


\[
\left( \forall x \exists y \right) \left( \forall z \exists w \right) Rxyzw
\]

\[
\forall z \quad \forall x \\
\quad \quad \quad \quad \downarrow \quad \downarrow \\
\exists w \quad \exists y
\]
**Domain** \( \{ 0, 1 \} \). \( \forall x \exists y \forall z \exists w Rxyzw \)
Domain \( \{0, 1\} \). \( (\forall x \exists y) \) \( R_{xyzw} \)
\begin{align*}
\begin{array}{cccc}
  x & y & z & w \\
  0 & 0 & 0 & 1 \\
  0 & 0 & 1 & 0 \\
  1 & 1 & 0 & 0 \\
  1 & 1 & 1 & 0 \\
\end{array}
\end{align*}

\textbf{AXIOMATIZATIONS}

\begin{align*}
\neg & = (z, w) \\
\models & = (z, w) \\
\end{align*}

\[
(\forall x \exists y) \quad Rxyzw \equiv \forall x \exists y \forall z \exists w (= (z, w) \land Rxyzw)
\]

**DEFINITION**

\textit{X a team} = set of assignments.
**Generalized quantifiers**

- **Generalized quantifier**: $Q$ a class of structures (one unary relation).
- $M, s \models Qx \phi$ iff $(M, \phi^M, s) \in Q$.
- $Q_M = \{ R \mid (M, R) \in Q \}$.
- $\text{FO}(Q)$ is FOL extended with expressions $Qx \phi$.
- $\exists$ and $\forall$ can be interpreted as generalized quantifiers.
- **We will only consider (non-trivial) monotone increasing quantifiers**: If $A \subseteq B$ and $A \in Q_M$ then $B \in Q_M$.
- $\tilde{Q} = \{ (M, R^c) \mid (M, R) \notin Q \}$. $\neg Qx \neg \phi \equiv \tilde{Q}x \phi$. 
**Dependence Logic with Q**

- $D(Q)$ is $\phi ::= \gamma \mid \phi \land \phi \mid \phi \lor \phi \mid \exists x \phi \mid \forall x \phi \mid Qx \phi$, where $\gamma$ is an FO($Q$) formula or dependence atom.

- $M \models \sigma$ iff $M, \{\emptyset\} \models \sigma$.

- $M, X \models \gamma$ if for all $s \in X$: $M, s \models \gamma$, where $\gamma$ is an FO($Q$) formula.

- $M, X \models \phi \land \psi$ iff $M, X \models \phi$ and $M, X \models \psi$.

- $M, X \models \phi \lor \psi$ iff there are $Y \cup Z = X$ such that $M, Y \models \phi$ and $M, Z \models \psi$. 
Quantifiers in Dependence Logic

- $M, X \models Qx \phi$ iff there is $F : X \rightarrow QM$ such that $M, X[F/x] \models \phi$.

$$X[F/x] = \{ s[a/x] \mid s \in X, a \in F(s) \}.$$ 

Example: $M, \{s_0, s_1\} \models Qz Rxyz$

- Note: using set-valued $Fs$ corresponds to non-deterministic strategies.
Properties of Dependence logic

- $M, \emptyset \models \phi$
- **Downwards closure**: If $Y \subseteq X$ and $M, X \models \phi$ then $M, Y \models \phi$.
- $D(Q) \equiv ESO(Q)$ (E / Kontinen)
- Branching of generalized quantifiers (p.o. quantifier prefixes) may be expressed in $D(Q)$.

**Theorem: Normal form for $D(Q, \tilde{Q})$**
Every $D(Q, \tilde{Q})$ formula is equivalent to one of the form:

$$
\mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m \exists y_1 \ldots \exists y_n \left( \bigwedge_{1 \leq i \leq n} = (\bar{x}^i, y_i) \land \theta \right),
$$

where $\mathcal{H}^i$ is either $Q, \tilde{Q}$ or $\forall$, and $\theta$ is a quantifier-free FO formula.
Axiomatizations
Dependence relations can be axiomatized (Armstrong).
Dependence logic has the same strength as ESO.
The relation $\Gamma \models \phi$ is not r.e.
Restricting to $\phi$’s without dependence atoms gives an r.e. entailment relation.
An explicit axiomatization has been given by Kontinen and Väänänen.

Idea:
Construct a natural deduction system in which the normal form can be derived.
Allow dependencies in normal forms to be replaced by finite approximations.
Show that in enough models (recursively saturated) the set of finite approximations is equivalent to the original sentence.
Axiomatizing $D(Q, \bar{Q})$ I: General rules

First: Some rules sound for any interpretation of $Q$ (monotone increasing).

- Standard rules for FO($Q, \bar{Q}$) formulas.
- Standard rules for conjunction, existential quantifier, and universal quantifier.
- Commutativity, associativity and monotonicity of disjunction.
- Monotonicity, extending scope, and renaming of bound variables for $Q$ and $\bar{Q}$.
- Duality of $\bar{Q}$ with respect to FO($Q, \bar{Q}$) formulas.
Axiomatizing $D(Q, \tilde{Q})$ II: Dependence related rules

- **Unnesting:**
  
  
  $\exists z(= (t_1, \ldots, z, \ldots, t_n) \land z = t_i)$

  where $z$ is a new variable.

- **Dependence distribution:**
  
  $\exists y_1 \ldots \exists y_n (\land_{1 \leq j \leq n} = (\tilde{z}_j, y_j) \land \phi) \lor \exists y_{n+1} \ldots \exists y_m (\land_{n+1 \leq j \leq m} = (\tilde{z}_j, y_j) \land \psi)$

  $\exists y_1 \ldots \exists y_m (\land_{1 \leq j \leq m} = (\tilde{z}_j, y_j) \land (\phi \lor \psi))$

  where $\phi$ and $\psi$ are quantifier free FO formulas.

- **Dependence introduction:**
  
  $\exists x \mathcal{H} y \phi$

  $\mathcal{H} y \exists x (= (\tilde{z}, x) \land \phi)$

  where $\tilde{z}$ lists the variables in $\text{FV}(\phi) - \{x, y\}$ and $\mathcal{H} \in \{\forall, Q, \tilde{Q}\}$. 
Approximations

Suppose \( \sigma \) is in normal form:

\[ \mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m \exists y_1 \ldots \exists y_n ( \bigwedge_{1 \leq i \leq n} = (\bar{x}^i, y_i) \land \theta(\bar{x}, \bar{y})) \].

Let \( A^k \sigma \) be

\[ \forall \bar{x}_1 \exists \bar{y}_1 \ldots \forall \bar{x}_k \exists \bar{y}_k ( \bigwedge_{1 \leq j \leq k} R(\bar{x}_j) \rightarrow \bigwedge_{1 \leq j \leq k} \theta(\bar{x}_j, \bar{y}_j) \land \bigwedge_{1 \leq i \leq n} (\bar{x}^i = \bar{x}^i_j \rightarrow y_{i,j} = y_{i,j'}) \) \]

Let \( B \sigma \) be

\[ \mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m R(x_1, \ldots, x_m) \].
Axiomatizing $D(Q, \tilde{Q})$ III: The approximation rule

\[ \begin{array}{c}
[B\sigma] & [A^k\sigma] \\
\vdots & \vdots \\
\sigma & \psi \\
\hline
\psi & \psi \quad \text{(Approx)}
\end{array} \]

where $\sigma$ is a sentence in normal form, and $R$ does not appear in $\psi$ nor in any uncancelled assumptions in the derivation of $\psi$, except for $B\sigma$ and $A^k\sigma$. 
Completeness for weak semantics

Let $\Gamma \models_w \phi$ mean that $\Gamma \models \phi$ for any monotone increasing (non-trivial) interpretation of $Q$ (and $\bar{Q}$ is interpreted as the dual of the interpretation of $Q$).

**Theorem**

This system is sound and complete wrt $\Gamma \models_w \phi$ where $\phi$ is FO($Q, \bar{Q}$).
UNCOUNTABLY MANY

- FO($Q_1$) is axiomatizable, where $Q_1$ is "there exist uncountably many ...". (Kiesler)
- Add Keisler’s rules for $Q_1$.

Define the **Skolem translation** $S\sigma$ of $\sigma$ in normal form to be:

$$H^1 x_1 \ldots H^n x_n \theta(f_i(x^i)/y_i).$$

- Replace the approximation rule with the following rule

$$[S\sigma]$$

* \[ \vdash \sigma \quad \psi \]

(Skolem)

**Theorem**

This system is sound and complete wrt $\Gamma \models \phi$ where $\phi$ is FO($Q_1, \bar{Q}_1$).
CONCLUSION

Extending dependence logic with generalized quantifiers is a natural and **stable** extension.

- The satisfaction relation is naturally defined when moving to non-deterministic strategies.
- \( D(Q) \) properly extends both \( \text{FO}(Q) \) and \( D \).
- \( D(Q) \) is equivalent to \( \text{ESO}(Q) \).
- \( D(Q, \tilde{Q}) \) has a prenex normal form theorem.
- Two completeness results:
  - First wrt to weak semantics.
  - Second wrt to \( Q_1 \).

- Second result not fully satisfactory. Is it possible to get completeness for \( Q_1 \) using approximations instead of Skolem functions?
Thank you for your attention.


