Implicitly definable generalized quantifiers

Filosofidagarna 2015, Linköping

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June 13, 2015

Generalized quantifiers

A **generalized quantifier** Q of type $\langle n_1, n_2, \ldots, n_k \rangle$ is a (class) function mapping sets to sets:

$$M \mapsto Q_M \subseteq \mathcal{P}(M^{n_1}) \times \mathcal{P}(M^{n_2}) \times \ldots \times \mathcal{P}(M^{n_k}).$$

For simplicity consider only generalized quantifiers of type $\langle 1 \rangle$:

$$Q_M \subseteq \mathcal{P}(M).$$

Syntax: $Qx\,\varphi$. Semantics:

$$M \vDash Qx\,\varphi \text{ iff } \varphi(M) \in Q_M$$

- $\forall_M = \{ M \}$
- $\exists_M = \{ A \subseteq M \mid A \neq \emptyset \}$
- $(Q_0)_M = \{ A \subseteq M \mid |A| \geq \aleph_0 \}$

Logicality

Logic considers the **form** of sentences and arguments. To determine this form we need to know which the **logical constants** are.

Which of the generalized quantifiers should be considered **logical**?

The ones that are **topic neutral**. (Ryle, 1954)

- 'Topic neutral' as 'not possible to discriminate between individuals' gives an **invariance** criterion.
- 'Topic neutral' as 'universally applicable' gives an **inferential** account.

The inferential viewpoint

Logicality is the property of being characterizable (uniquely) by inference rules.

Thus, the meaning of conjunction is given by the rules:

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad \frac{\varphi \wedge \psi}{\varphi} \qquad \frac{\varphi \wedge \psi}{\psi}$$

Uniqueness: Introduce two new symbols $\wedge_1$ $\wedge_2$:

$$\frac{\varphi \quad \psi}{\varphi \wedge_1 \psi} \qquad \frac{\varphi \wedge_1 \psi}{\varphi} \qquad \frac{\varphi \wedge_1 \psi}{\psi}$$

$$\frac{\varphi \quad \psi}{\varphi \wedge_2 \psi} \qquad \frac{\varphi \wedge_2 \psi}{\varphi} \qquad \frac{\varphi \wedge_2 \psi}{\psi}$$

Then $\varphi \wedge_1 \psi \dashv\vdash \varphi \wedge_2 \psi$.

Feferman's approach

Let L_2 be **pure second order logic**:

- Individual variables: $x, y, z, \ldots,$
- Predicate variables (including 0-ary) $P, P_1, \ldots$
- Formulas are built from predicate variables using $\neg, \vee, \wedge, \rightarrow, \forall, \exists$.

Semantics is **Henkin semantics**:

- A model M of L_2 is a pair of a set M and a set $\mathrm{Pred}(M)$ of subsets of $\mathcal{P}(M^k)$, $k \geq 1$, for the predicate variables to range over.

Definability

- The language $L_2(\mathsf{Q})$ is L_2 extended with a **second-order** predicate symbol Q. Example: $\forall P\,\mathsf{Q}(P)$.
- A model of $L_2(\mathsf{Q})$ gives an interpretation for Q as a second-order predicate, i.e., a subset of $\mathrm{Pred}(M)$.
- We say that a sentence θ of $L_2(\mathsf{Q})$ **implicitly** defines a generalized quantifier Q if for every L_2 model M the only second-order predicate satisfying θ is $Q_M \cap \mathrm{Pred}(M)$.
- A formula $\theta(P)$ of L_2 **explicitly** defines a generalized quantifier Q if for every L_2 model M, for every $R \in \mathrm{Pred}(M)$:

$$(M, R) \vDash \theta(P) \text{ iff } R \in Q_M.$$

Logicality

According to Feferman's (new) thesis on logicality:

A generalized quantifier Q is **logical** iff
it is implicitly definable in L_2.

Main Theorem (Feferman)

Q is implicitly definable in L_2 iff it is (explicitly) definable in FOL.

Proof of the Main Theorem

Beth's theorem

Suppose first-order logic. If

$$T, \sigma(P), \sigma(P') \vDash \forall \bar{x}(P\bar{x} \leftrightarrow P'\bar{x})$$

then there is a formula $\varphi(\bar{x})$ (without P) such that

$$T, \sigma(P) \vDash \forall \bar{x}(P\bar{x} \leftrightarrow \varphi(\bar{x})).$$

Proof of the Main theorem is by:

- translating to many-sorted first-order logic,
- then using Beth's theorem for many-sorted formulas (proved by Feferman in 1968) and
- then argue that the many-sorted formula explicitly defining Q is equivalent to a first-order formula defining Q.

Alternative proof of the main Theorem

Suppose Q of type $\langle 1 \rangle$ is implicitly defined by θ.
Fix a universe M and for every $A \subseteq M$ let

$$M_A = (M, \{ A \})$$

be the L_2 model in which the predicate variables range over the singleton set $\{ A \}$.
θ may not include n-ary predicate symbols for $n \geq 2$.
Let $Q'_M = \mathcal{P}(M)$ be the universally true second order predicate.
Then $(M_A, Q'_M) \vDash \theta$ iff $Q'_M \cap \{ A \} = Q_M \cap \{ A \}$ iff $A \in Q_M$.
Let φ be the first-order formula we get from θ by removing second-order quantifiers and replacing all predicate variables by the single predicate variable P. Also repacing all $\mathsf{Q}(P)$ by $\top$. Then

$$(M, A) \vDash \varphi \text{ iff } (M_A, Q'_M) \vDash \theta \text{ iff } A \in Q_M$$

and thus φ defines Q.

Conclusions ...

The main theorem says that "plugging in" pure second-order logic into the machinery gives us first-order logic back, i.e.,

$$\mathrm{Beth}^2(L_2, \mathrm{FOL}).$$

However, this argument shows that this is for completetely elementary reasons:

Pure second-order logic with Henkin semantics "is" just first-order logic.

...and question

- In fact, the grounds for considering Henkin semantics are not clear.
- Also, we may observe that many inference rules can be formalized by a Π^1_1 formula.

Which quantifiers are implicitly definable in full second-order logic (i.e., second-order logic with standard semantics) with a Π^1_1 sentence?

THANK YOU!

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