Implicitly definable
Generalized Quantifiers

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**Generalized Quantifiers**

A **generalized quantifier** $Q$ of type $\langle n_1, n_2, \ldots, n_k \rangle$ is a (class) function mapping sets to sets:

$$M \mapsto Q_M \subseteq \mathcal{P}(M^{n_1}) \times \mathcal{P}(M^{n_2}) \times \ldots \times \mathcal{P}(M^{n_k}).$$

For simplicity consider only generalized quantifiers of type $\langle 1 \rangle$:

$$Q_M \subseteq \mathcal{P}(M).$$

**Syntax:** $Qx \varphi$. **Semantics:**

$$M \models Qx \varphi \text{ iff } \varphi(M) \in Q_M$$

- $\forall_M = \{ M \}$
- $\exists_M = \{ A \subseteq M \mid A \neq \emptyset \}$
- $(Q_0)_M = \{ A \subseteq M \mid |A| \geq \aleph_0 \}$
**Logicality**

Logic considers the **form** of sentences and arguments. To determine this form we need to know which the **logical constants** are.

Which of the generalized quantifiers should be considered **logical**?

The ones that are **topic neutral**. (Ryle, 1954)

- ‘Topic neutral’ as ‘not possible to discriminate between individuals’ gives an **invariance** criterion.
- ‘Topic neutral’ as ‘universally applicable’ gives an **inferential** account.
**The inferential viewpoint**

Logicality is the property of being characterizable (uniquely) by inference rules.
Thus, the meaning of conjunction is given by the rules:

\[
\begin{align*}
\varphi & \quad \psi \\
\hline 
\varphi \wedge \psi & \\
\end{align*}
\]

Uniqueness: Introduce two new symbols \( \wedge_1 \wedge_2 \):

\[
\begin{align*}
\varphi & \quad \psi \\
\hline 
\varphi \wedge_1 \psi & \\
\varphi \wedge_2 \psi & \\
\end{align*}
\]

Then \( \varphi \wedge_1 \psi \vdash \varphi \wedge_2 \psi \).
Let $L_2$ be **pure second order logic**:

- Individual variables: $x, y, z, \ldots$
- Predicate variables (including 0-ary) $P, P_1, \ldots$
- Formulas are built from predicate variables using $\neg, \lor, \land, \rightarrow, \forall, \exists$.

Semantics is **Henkin semantics**:

- A model $M$ of $L_2$ is a pair of a set $M$ and a set $\text{Pred}(M)$ of subsets of $\mathcal{P}(M^k)$, $k \geq 1$, for the predicate variables to range over.
Definability

- The language $L_2(Q)$ is $L_2$ extended with a second-order predicate symbol $Q$. Example: $\forall P Q(P)$.
- A model of $L_2(Q)$ gives an interpretation for $Q$ as a second-order predicate, i.e., a subset of $\text{Pred}(M)$.
- We say that a sentence $\theta$ of $L_2(Q)$ implicitly defines a generalized quantifier $Q$ if for every $L_2$ model $M$ the only second-order predicate satisfying $\theta$ is $Q_M \cap \text{Pred}(M)$.
- A formula $\theta(P)$ of $L_2$ explicitly defines a generalized quantifier $Q$ if for every $L_2$ model $M$, for every $R \in \text{Pred}(M)$:

  $$(M, R) \models \theta(P) \text{ iff } R \in Q_M.$$
According to Feferman’s (new) thesis on logicality:

A generalized quantifier $Q$ is **logical** iff it is implicitly definable in $L_2$.

**Main Theorem (Feferman)**

$Q$ is implicitly definable in $L_2$ iff it is (explicitly) definable in FOL.
Proof of the Main Theorem

Beth’s theorem

Suppose first-order logic. If

\[ T, \sigma(P), \sigma(P') \models \forall \bar{x}(Px \leftrightarrow P'\bar{x}) \]

then there is a formula \( \varphi(\bar{x}) \) (without \( P \)) such that

\[ T, \sigma(P) \models \forall \bar{x}(Px \leftrightarrow \varphi(\bar{x})). \]

Proof of the Main theorem is by:

- translating to many-sorted first-order logic,
- then using Beth’s theorem for many-sorted formulas (proved by Feferman in 1968) and
- then argue that the many-sorted formula explicitly defining \( Q \) is equivalent to a first-order formula defining \( Q \).
ALTERNATIVE proof of the main Theorem

Suppose $Q$ of type $\langle 1 \rangle$ is implicitly defined by $\theta$.
Fix a universe $M$ and for every $A \subseteq M$ let

$$M_A = (M, \{ A \})$$

be the $L_2$ model in which the predicate variables range over the singleton set $\{ A \}$.
$\theta$ may not include $n$-ary predicate symbols for $n \geq 2$.
Let $Q'_M = \mathcal{P}(M)$ be the universally true second order predicate.
Then $(M_A, Q'_M) \models \theta$ iff $Q'_M \cap \{ A \} = Q_M \cap \{ A \}$ iff $A \in Q_M$.
Let $\varphi$ be the first-order formula we get from $\theta$ by removing second-order quantifiers and replacing all predicate variables by the single predicate variable $P$. Also repacing all $Q(P)$ by $\top$. Then

$$(M, A) \models \varphi \text{ iff } (M_A, Q'_M) \models \theta \text{ iff } A \in Q_M$$

and thus $\varphi$ defines $Q$. 
CONCLUSIONS ...

The main theorem says that “plugging in” pure second-order logic into the machinery gives us first-order logic back, i.e.,

\[ \text{Beth}^2(L_2, \text{FOL}). \]

However, this argument shows that this is for completely elementary reasons:

Pure second-order logic with Henkin semantics “is” just first-order logic.
...AND QUESTION

- In fact, the grounds for considering Henkin semantics are not clear.
- Also, we may observe that many inference rules can be formalized by a $\Pi_1^1$ formula.

Which quantifiers are implicitly definable in full second-order logic (i.e., second-order logic with standard semantics) with a $\Pi_1^1$ sentence?
THANK YOU!
Fredrik Engström.  
Implicitly definable generalized quantifiers.  

Solomon Feferman.  
Which quantifiers are logical? a combined semantical and inferential criterion.  
2012.