Non permutation invariant Borel quantifiers

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Logical constants

An operator is to be counted as a logical constant if it is **topic neutral**.

- Examples: $\exists$, $\forall$, $\neg$, and $\rightarrow$.
- Non-examples: “There is a straight line such that. . . ”

Mautner, Tarski, Mostowski, Lindenbaum: Logic is the the study of the invariants under the most general transformations (=permutations). (cf. Klein’s Erlangen program)

Generalized quantifers

Definition (Mostowski)

A **generalized quantifier** *Q of type $\langle n \rangle$ is a class of structures in the language $\{ R \}$ where R is an n-ary relation symbol.*

Examples:

- $\exists = \{ (M, A) \mid A \subseteq M, A \neq \emptyset \}$
- $\forall = \{ (M, M) \mid M \}$
- $Q_0 = \{ (M, A) \mid A \subseteq M, |A| \geq \aleph_0 \}$
- $W = \{ (M, R) \mid R \subseteq M^2, R \text{ is well-founded} \}$
- $Q^A = \{ (M, B) \mid A \subseteq B \}$ (principal)

- $\varphi(M) = \{ \bar{a} \in M^k \mid M \models \varphi(\bar{a}) \}$
- $M \models Q\bar{x}\ \varphi(\bar{x})$ *iff* $(M, \varphi(M)) \in Q$

Generalized quantifers contd.

Local versions: For a given domain M, let

$$Q_M = \left\{ R \subseteq M^k \;\middle|\; (M, R) \in Q \right\}.$$

Definition

A (local) quantifier Q_M is **definable** *on the domain M in the logic L if there is φ of L, such that*

$$(M, R) \models \varphi \text{ iff } R \in Q_M.$$

Fix a domain M.
Quantifier now means **local** quantifier on M.

Logicality as invariance

Definition

Q is **fixed by/invariant under** *a permutation g of M if* $g(Q) = Q$*, where* $g(Q) = \{ g(R) \mid R \in Q \}$.

Tarski's thesis

A local quantifier on a domain M is a logical constant iff it is invariant under all **permutations** *of M.*

Examples: $\exists, \forall, Q_0$
Non-examples: Q^A

Theorem (McGee -91 / Krasner -38)

Q is permutation invariant iff there is a formula in $\mathscr{L}_{\infty\infty}$ *defining Q on M.*

Infinitary syntax

$\mathscr{L}_{\infty\infty}$ is FOL with the infinitary operations:

- $\bigvee \Phi$, *and*
- $\exists X \varphi$,

where Φ is any **set** *of formulas and X any* **set** *of variables.*

With the natural semantics attached.

$\mathscr{L}_{\omega_1\omega}$ is FOL with the infinitary operation:

- $\bigvee \Phi$,

where Φ is a countable set of formulas with at most finitely many free variables.

A Galois connection

$\mathscr{Q}$ is a set of quantifiers. G subgroup of the full symmetric group on M.

Definition

- *Let* Aut($\mathscr{Q}$) *be the group of all permutations of M fixing all quantifiers in $\mathscr{Q}$:*
  $\mathrm{Aut}(\mathscr{Q}) = \{ g \in Sym(M) \mid g(Q) = Q \text{ for all } Q \in \mathscr{Q} \}$.
- *Let* Inv(G) *be the set of quantifiers fixed by all permutations in G:* $\mathrm{Inv}(G) = \{ Q \mid g(Q) = Q \text{ for all } g \in G \}$.

Theorem (Krasner/Bonnay/E)

- $\mathrm{Aut}(\mathrm{Inv}(G)) = G$
- $\mathrm{Inv}(\mathrm{Aut}(\mathscr{Q}))$ *is the set of quantifiers definable in* $\mathscr{L}_{\infty\infty}(\mathscr{Q})$

The Borel hierarchy

Given relational lexicon τ, X_τ is the set of all τ structures on $\mathbb{N}$, a subset of X_τ is **open** if it is the union of sets of the form

$$\{ M \in X_\tau \mid M \models R_1(\bar{a}_1) \wedge \ldots \wedge R_k(\bar{a}_k) \} .$$

The **Borel sets** are built up from the open sets by complementation and countable unions:
Σ^B_0 = open sets.
Π^B_α = complements of Σ^B_α sets.

$$\Sigma^B_\alpha = \left\{ \bigcup_{i \in \mathbb{N}} A_i \;\middle|\; A_i \in \Pi^B_{\alpha_i}, \alpha_i < \alpha \right\}$$

The set of Borel sets is the union of all Σ^B_α for $\alpha \in \mathrm{Ord}$.
Borel sets are **predicative** and **set theoretic absolute**. Thus, a natural condition to put on logical constants.

Borel quantifiers

Assume now all quantifiers are local quantifiers on $\mathbb{N}$. Thus a quantifier is nothing but a **subset of X_τ**.

Theorem (Lopez-Escobar (-65))

A quantifier is **Borel** *and* **permutation invariant** *iff it is definable in $\mathscr{L}_{\omega_1\omega}$.*

Indicates a strong connection between $\mathscr{L}_{\omega_1\omega}$ and Borel quantifiers. Does the Lopez-Escobar theorem generalize to $\mathscr{L}_{\omega_1\omega}(Q)$ as the result for $\mathscr{L}_{\infty\infty}$ does?

Non permutation invariant Borel quantifiers

FALSE

Q is Borel and $\mathrm{Aut}(\mathscr{Q})$ invariant iff Q is definable in $\mathscr{L}_{\omega_1\omega}(\mathscr{Q})$.

Let $A \subseteq \mathbb{N}$ be infinite and coinfinite and $Q' = \{ A \}$ then Q^A is $\mathrm{Aut}(Q')$ invariant, but not definable in $\mathscr{L}_{\omega_1\omega}(Q')$.

The **orbit** of $\bar{a}$ under G is $\{ g(\bar{a}) \mid g \in G \}$.

Generalizations of Lopez-Escobar

Observation: Vaught's proof (-74) of the Lopez-Escobar theorem generalizes to:

Proposition

Suppose $G \leq Sym(\mathbb{N})$ is **closed** *and $\mathscr{F}$ is the family of G-orbits of tuples. A subset of X_τ is Borel and G-invariant iff it is definable in $\mathscr{L}_{\omega_1\omega}(\mathscr{F})$.*

Since the automorphism group of any set of relations is closed we have the following.

Proposition

Every $Aut(\mathscr{R})$-invariant Borel subset of X_τ is definable in $\mathscr{L}_{\omega_1\omega}(\mathscr{R})$.

What about quantifiers?

It follows that if Aut(Q) **is closed** and **its orbits are definable in** $\mathscr{L}_{\omega_1\omega}(Q)$ then a subset of X_τ is Borel and Aut(Q)-invariant iff it is $\mathscr{L}_{\omega_1\omega}(Q)$ definable.

Theorem

If Q is either

- **good** *(a technical definition),*
- **clopen**, *or*
- **a finite boolean combination of Q^As**,

then a subset of X_τ is Borel and Aut(Q)*-invariant iff it is* $\mathscr{L}_{\omega_1\omega}(Q)$ *definable.*

These three classes are subclasses of the Borel quantifiers. $\exists$, $\forall$, Q_0 and Q^A are all good.
In other words: BInv(Aut(Q)) is the $\mathscr{L}_{\omega_1\omega}$-closure of $\{ Q \}$.

The other way

Every subgroup G is $\mathrm{Aut}(Q)$ for some quantifier Q.

Theorem

Every **closed** *subgroup G is $\mathrm{Aut}(Q)$ for some* **good** *quantifier Q.*

Thus $\mathrm{Aut}(\mathrm{BInv}(G)) = G$ for every closed subgroup G:

$$\mathrm{Aut}(\mathrm{BInv}(G)) = \mathrm{Aut}(\mathrm{BInv}(\mathrm{Aut}(Q))) = \mathrm{Aut}(Q) = G$$

Thanks