INVARINACE AND DEFINABILITY, WITH OR WITHOUT EQUALITY

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Fredrik Engström, Göteborg
Joint work with Denis Bonnay, Paris

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INTRODUCTION
**Invariance**

- **Klein’s Erlangen Program**: Invariance as the defining property for geometries.
- **Tarski’s thesis**: Extend to logics; use invariance as defining property for logics and logical operators. (Tarski, 1986)
- **Idea**: Extend the correspondence of invariance and operators to a full Galois connection: $\text{Inv}$ maps invariance criteria to sets of operators, and $\text{Aut}$ maps sets of operators to invariance criteria such that

  $$\mathcal{D} \subseteq \text{Inv}(H) \iff H \subseteq \text{Aut}(\mathcal{D}), \text{ and}$$

  $\text{Inv} (\text{Aut}(\mathcal{D}))$ corresponds to definability in a logic $L$. 

Quantifiers

Definition (Mostowski/Lindström)

A generalized quantifier $Q$ of type $\langle n_1, \ldots, n_k \rangle$ is a (class) of structures in the language $\{ R_1, \ldots, R_k \}$ where $R_i$ is of arity $n_i$.

Examples:

- $\exists = \{ (M, A) \mid A \subseteq M, A \neq \emptyset \}$
- $\forall = \{ (M, M) \mid M \}$
- $Q_0 = \{ (M, A) \mid A \subseteq M, |A| \geq \aleph_0 \}$

Definition

$M \models Q\overline{x}\varphi(\overline{x})$ iff $(M, R) \in Q$, where $R = \{ \overline{a} \in \Omega^k \mid M \models \varphi(\overline{a}) \}$.

- **Local** quantifier: $Q_\Omega = \{ \langle R_1, \ldots, R_k \rangle \mid (\Omega, R_1, \ldots, R_k) \in Q \}$

- A local quantifier, of type $\langle n \rangle$, is definable on $\Omega$ in the logic $\mathcal{L}$ if there is $\varphi$ of $\mathcal{L}$, such that $(\Omega, R) \models \varphi$ iff $R \in Q$. 
Galois theory:
\[ \{ H \subseteq \text{Aut}(K : k) \} \implies \{ A \mid k \subseteq A \subseteq K \} \]
least group least field

Krasner’s Galois theory:
\[ \{ H \subseteq \text{Sym}(\Omega) \} \implies \{ M \text{ infinitary rel. structure on } \Omega \} \]
least group definability in \( \mathcal{L}_{\infty\infty} \)

Our results:
\[ \{ H \subseteq \text{Sym}(\Omega) \} \implies \{ \mathcal{Q} \text{ set of quantifiers on } \Omega \} \]
definability in \( \mathcal{L}_{\infty\infty} \)
\[ \{ \Pi \text{ set of similarities on } \Omega \} \implies \{ \mathcal{Q} \text{ set of quantifiers on } \Omega \} \]
definability* in \( \mathcal{L}_{\infty\infty} \)
Motivation I

Tarski’s thesis on logicality (Tarski, 1986)
A (local) quantifier on a domain $\Omega$ is a logical constant iff it is invariant under all permutations of $\Omega$.

McGee’s Theorem (McGee, 1996)
A local quantifier $Q$ on $\Omega$ is permutation invariant iff it is $\mathcal{L}_{\infty\infty}$-definable.

- Galois connection results give stronger connections between logics and invariance criteria: The connections are stable under adding operations.
Motivation II

Monadic quantifier: Quantifiers of type $\langle 1, \ldots, 1 \rangle$.

**Feferman’s thesis on logicality (Feferman, 1999)**

A quantifier is a logical constant iff it can be defined (in typed $\lambda$-calculus) from equality and monadic quantifiers invariant under talking preimages of surjections.

**Feferman’s theorem (Feferman, 1999)**

Monadic quantifiers are invariant under preimages of surjections iff they are definable in $L^-_{\omega \omega}$.

- Feferman leaves the general question for arbitrary quantifiers open.
- Our result on the equality-free version of $L_{\omega \omega}$ is a variant on Feferman’s theorem, generalized to a full Galois connection.
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**With equality**
A Galois connection

- Fix a domain $\Omega$. Quantifier means local quantifier on $\Omega$.
- $\mathcal{Q}$ is a set of quantifiers.
- $G$ subgroup of the full symmetric group $\text{Sym}(\Omega)$.

**Definition**

- Let $\text{Aut}(\mathcal{Q})$ be the group of all permutations of $\Omega$ fixing all quantifiers in $\mathcal{Q}$:
  $$\text{Aut}(\mathcal{Q}) = \{ g \in \text{Sym}(\Omega) \mid g(Q) = Q \text{ for all } Q \in \mathcal{Q} \}.$$  
- Let $\text{Inv}(G)$ be the set of quantifiers fixed by $G$:
  $$\text{Inv}(G) = \{ Q \mid g(Q) = Q \text{ for all } g \in G \}.$$  

**Theorem (Krasner, 1938, 1950), (B/E)**

- $\text{Aut}(\text{Inv}(G)) = G$
- $\text{Inv}(\text{Aut}(\mathcal{Q}))$ is the set of quantifiers definable in $\mathcal{L}_{\infty\infty}(\mathcal{Q})$

There is a permutation group which is not $\text{Aut}(\mathcal{Q})$ for any set of monodic quantifiers $\mathcal{Q}$. 
**Proof**

**Aut(Inv(G)) = G:** Let $\leq$ well-order $\Omega$, and $Q = \{ g(\leq) \mid g \in G \}$ of type $\langle 2 \rangle$. If $h \in \text{Aut}(\text{Inv}(G))$ then $h(\leq) \in Q$ and so there is $g \in G$ such that $h(\leq) = g(\leq)$, implying $h = g$.

**Inv(Aut(Ω)) is the set of Qs definable in \(L_{\infty\infty}(Ω)\):** We assume all quantifiers of type $\langle 1 \rangle$ and $\Omega = \omega$. $Q' \in \text{Inv}(\text{Aut}(Ω))$ is defined by

$$\forall x_0, x_1, \ldots \left[ \bigwedge_{i \neq j} x_i \neq x_j \land \bigwedge_{y \in i} y = x_i \land \bigwedge_{Q \in Ω, A \in Q, i \in A} \left( (\bigwedge_{y = x_i} y) \land (\bigwedge_{A \notin Q, i \in A} \neg y = x_i) \right) \rightarrow \bigvee_{A \in Q', i \in A} \left( \bigwedge_{x_i} P \land \bigwedge_{i \notin A} \neg P \right) \right]$$
Without equality
Plan

- We want a Galois connection involving the equality free logic $\mathcal{L}_{\infty\infty}$.
- **Idea:** Work in $\Omega/\sim$, where $\sim$ is the finest definable equivalence relation and apply the previous result.
- **Problem:** Can we define $\sim$ without knowing the language?
- **Solution:** Yes... sometimes.


**Definitions**

- $\pi$ is a **similarity relation** on $\Omega$ if $\text{dom}(\pi) = \text{rng}(\pi) = \Omega$.
- $R \pi S$ if $\forall \bar{a}, \bar{b} \in \Omega$ such that $\bar{a} \pi \bar{b}$: $\bar{a} \in R$ iff $\bar{b} \in S$.
- $R$ is **invariant** under $\pi$ if $R \pi R$.

Invariance for quantifiers is parametrized by an equivalence relation:

**Definition**

A quantifier $Q$ on $\Omega$ is **$\sim$-invariant** under $\pi$ if for all relations $R_1, \ldots, R_k$, $S_1, \ldots, S_k$ on $\Omega$ **invariant under** $\sim$ such that $R_i \pi S_i$ we have $\langle R_1, \ldots, R_k \rangle \in Q$ iff $\langle S_1, \ldots, S_k \rangle \in Q$.

Motivation: The language $\mathcal{L}_{\infty\infty}(\mathcal{D})$ can be very restricted: we can only talk about the **definable** sets/relations.
The mappings

- A set of operations $\mathcal{D}$ generates an equivalence relation $\sim_{\mathcal{D}}$, the finest $L_{\infty\infty}(\mathcal{D})$-definable equivalence relation.

- Dually, a set of similarities $\Pi$ gives us an equivalence relation by the following condition:

  $a \approx_{\Pi} b$ if for all $\bar{c} \in \Omega^k$ there is $\pi \in \Pi$ such that $a, \bar{c} \pi b, \bar{c}$.

The mappings for the Galois connection can now be defined:

- $\text{Sim}(\mathcal{D})$ is the set of similarities $\pi$ such that all relations and quantifiers in $\mathcal{D}$ are $\sim_{\mathcal{D}}$-invariant under $\pi$.

- $\text{Inv}(\Pi)$ is the set of all relations $R$ and quantifiers $Q$ on $\Omega$ which are $\approx_{\Pi}$-invariant under all similarities in $\Pi$. 
First half of the correspondence

Let the blow-up $\hat{Q}$ of $Q$ relative to $\sim$ be $\{\hat{R} \mid R \in Q\}$, where

$$\hat{R} = \{ \langle a_1, \ldots, a_k \rangle \mid \exists \langle b_1, \ldots, b_k \rangle \in R, a_1 \sim b_1, \ldots a_k \sim b_k \}.$$ 

**Theorem**

Let $\mathcal{D}$ be a set of operators then

1. $Q \in \text{Inv}(\text{Sim}(\mathcal{D}))$ iff $\hat{Q}$ is definable in $L_{\infty}\negation{\infty}(\mathcal{D})$.
2. $R \in \text{Inv}(\text{Sim}(\mathcal{D}))$ iff $R$ is definable in $L_{\infty}\negation{\infty}(\mathcal{D})$. 
More definitions

- A similarity $\pi$ is **identity-like** (with respect to $\Pi$) if $\pi \subseteq \approx_{\Pi}$.
- A set $\Pi$ of similarities is **saturated** if it includes all identity-like similarities.
- $\Pi$ is a **monoid with involution** if it is closed under composition and taking converses.
- $\Pi$ is **full** if it is a saturated monoid with involution closed under taking subsimilarities, i.e., such that if $\pi \in \Pi$ and $\pi' \subseteq \pi$ is a similarity then $\pi' \in \Pi$.

**Theorem**

Let $\Pi$ be a set of similarity relations, then $\Sim(\Inv(\Pi))$ is the smallest full monoid including $\Pi$. 
Thank you for your attention.
Bibliography


