Dependence logic with generalized quantifiers

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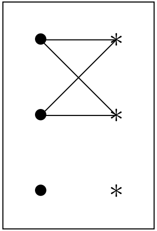
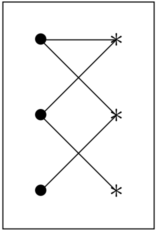
Introduction

Branching in natural languages

Most of the dots and most of the stars are all connected by lines. (Barwise, 1979)



$$\begin{pmatrix} Q_1 x \\ Q_2 y \end{pmatrix} R(x, y)$$



$$Q_1 x\, Q_2 y\, R(x, y)$$
$$Q_2 y\, Q_1 x\, R(x, y)$$

The semantics of branching

- A generalized quantifier Q is a class of structures.
- $Q_M = \{ \bar{R} \mid (M, \bar{R}) \in Q \}$.
- $M \models Qx\varphi$ iff $\varphi^M \in Q_M$.
- Q_1 and Q_2 are **increasingly monotone** quantifiers.

$$\begin{pmatrix} Q_1 x \\ Q_2 y \end{pmatrix} R(x, y)$$

is defined as

$$\mathrm{Br}(Q_1, Q_2)xy\, R(x, y).$$

$$\mathrm{Br}(Q_1, Q_2)_M = \left\{ R \subseteq M^2 \;\middle|\; \exists A \in Q_{1M}\, \exists B \in Q_{2M} : A \times B \subseteq R \right\}.$$

- Not compositional!

Dependence Logic

Dependence logic

- Dependence logic: FOL + $\mathrm{D}(t_1, \ldots, t_k)$ (Väänänen, 2007)
- (Negation may only appear in front of atomic formulas.)
- The Henkin quantifier $\begin{pmatrix} \forall x \exists y \\ \forall z \exists w \end{pmatrix}$ corresponds to the formula:

$$\forall x \exists y \forall z \exists w (\mathrm{D}(z, w) \wedge \ldots)$$

Hodges' semantics

- X is a **team**, i.e., a set of assignments.
- $M, X \models \varphi$.
- For first-order φ: $M, X \models \varphi$ iff for all $s \in X$: $M, s \models \varphi$.
- $M, X \models \neg\, \mathrm{D}(\bar{x})$ iff $X = \emptyset$.

$$M, X \models \mathrm{D}(\bar{x}, y)$$

iff for all $s, s' \in X$ if $s(\bar{x}) = s'(\bar{x})$ then $s(z) = s'(z)$.

x	y	z
1	4	4
1	5	4
2	4	2
2	6	2

- $M, X \not\models x = z$
- $M, X \not\models x \neq z$
- $M, X \models \mathrm{D}(x, z)$
- $M, X \not\models \mathrm{D}(x, y)$

Hodges' semantics II

- $M, X \vDash \varphi \wedge \psi$ iff $M, X \vDash \varphi$ and $M, X \vDash \psi$.
- $M, X \vDash \varphi \vee \psi$ iff there are Y and Z such that $M, Y \vDash \varphi$ and $M, Z \vDash \psi$ and $X = Y \cup Z$.
- $M, X \vDash \exists x \varphi$ iff there is $f : X \to M$ such that $M, X[f/x] \vDash \varphi$.
- $M, X \vDash \forall x \varphi$ iff $M, X[M/x] \vDash \varphi$.

$$X[f/x] = \{ s[f(s)/x] \mid s \in X \}.$$

$$X[M/x] = \{ s[a/x] \mid a \in M, s \in X \}.$$

- $M \vDash \sigma$ iff $M, \{ \epsilon \} \vDash \sigma$.

Branching in Dependence Logic

$$M \models \mathrm{Br}(\forall\exists, \forall\exists)xyzw\, R(x, y, z, w)$$

iff

$$M \models \forall x \exists y \forall z \exists w \left(\mathrm{D}(z, w) \wedge R(x, y, z, w)\right)$$

What about generalized quantifiers?

$$M \models \mathrm{Br}(Q_1, Q_2)xy\, R(x, y)$$

iff

$$M \models Q_1 x\, Q_2 y \left(\mathrm{D}(y) \wedge R(x, y)\right)$$

?

Generalized quantifiers in Dependence logic

Lifting functions

The **Hodges space** of order ideals on the power set is

$$\mathcal{H}(A) = \mathcal{L}(\mathcal{P}(A)).$$

Given $h : \mathcal{P}(A) \to \mathcal{P}(B)$ we define the **lift**:

$$\mathcal{L}(h) : \mathcal{H}(A) \to \mathcal{H}(B), \mathscr{X} \mapsto \downarrow\{\, h(X) \mid X \in \mathscr{X} \,\},$$

where $\downarrow\mathscr{X}$ is the downward closure of $\mathscr{X}$, i.e.

$$\downarrow\mathscr{X} = \{\, X \mid \exists Y \in \mathscr{X}, X \subseteq Y \,\}.$$

Lifting quantifers

- Q a monotone type $\langle 1 \rangle$ quantifier.
- $Q_M : \mathcal{P}(M^{n+1}) \to \mathcal{P}(M^n)$
- $\mathcal{L}(Q_M) : \mathcal{H}(M^{n+1}) \to \mathcal{H}(M^n)$
- Gives truth conditions for Q in Hodges semantics:

  $M, X \vDash Qx\varphi$ iff there is $F : X \to Q_M$ such that $M, X[F/x] \vDash \varphi$.

  where $X[F/x] = \{ s[a/x] \mid a \in F(s), s \in X \}$.
- $\mathcal{L}$ applied to $\exists$ and $\forall$ give equivalent truth conditions for $\exists$ and $\forall$.

Proposition

For $\mathrm{FO}(Q)$ formulas φ:

$$M, X \vDash \varphi \text{ iff for all } s \in X : \; M, s \vDash \varphi.$$

Generalized quantifiers and dependence atoms

If Q_M contains no singletons and $X \neq \emptyset$ then $M, X \not\models Qx\,(\mathrm{D}(x) \wedge \varphi)$.

$$M \models \mathrm{Br}(Q_1, Q_2)xy\,R(x, y)$$

iff

$$M \models Q_1 x\, Q_2 y\, \big(\mathrm{D}(y) \wedge R(x, y)\big)$$

Dependence Logic with GQ

Proposition (Engström and Kontinen)

For non-trivial Q, $D(Q) \equiv \mathrm{ESO}(Q)$.

- Thus, $D(\mathrm{Br}(Q_1, Q_2)) \leq D(Q_1, Q_2)$ and so branching of generalized quantifiers can be defined with the dependence atom.
- Open question: Can this be done compositionally?

Multivalued Dependence

Multivalued dependence and teams

Definition

$$M, X \models [\bar{x} \twoheadrightarrow y] \text{ if}$$

for all $s, s' \in X$ such that $s(\bar{x}) = s'(\bar{x})$ there exists $s_0 \in X$ such that $s_0(\bar{x}) = s(\bar{x})$, $s_0(y) = s(y)$, and $s_0(\bar{z}) = s'(\bar{z})$, where $\bar{z}$ are the variables in $\mathrm{dom}(X) \setminus (\{\, \bar{x} \,\} \cup \{\, y \,\})$.

Proposition

If Q_1 and Q_2 are monotone then $M \models \mathrm{Br}(Q_1, Q_2)xy\, R(x, y)$ iff

$$M \models Q_1 x\, Q_2 y \left([\twoheadrightarrow y] \wedge R(x, y)\right).$$

Proposition

FOL + multivalued dependence $\equiv D$.

Embedded multivalued dependence

- Multivalued dependence is **dependent on context**.

Definition

$M, X \models [\bar{x} \twoheadrightarrow \bar{y} \mid \bar{z}]$ if
$Y \models [\bar{x} \twoheadrightarrow \bar{y}]$ where Y is the projection of X onto $\{ \bar{x}, \bar{y}, \bar{z} \}$.

- $[\bar{x} \twoheadrightarrow \bar{y} \mid \bar{z}]$ is **independent on context**.
- This is the **independence atom** introduced by Väänänen and Grädel: $\bar{y} \perp_{\bar{x}} \bar{z}$ iff $[\bar{x} \twoheadrightarrow \bar{y} \mid \bar{z}]$
- However, embedded multivalued dependence is **not** axiomatizable. (Sagiv and Walecka, 1982) (Both functional and multivalued dependence are.)

Thank you for your attention.

Bibliography

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